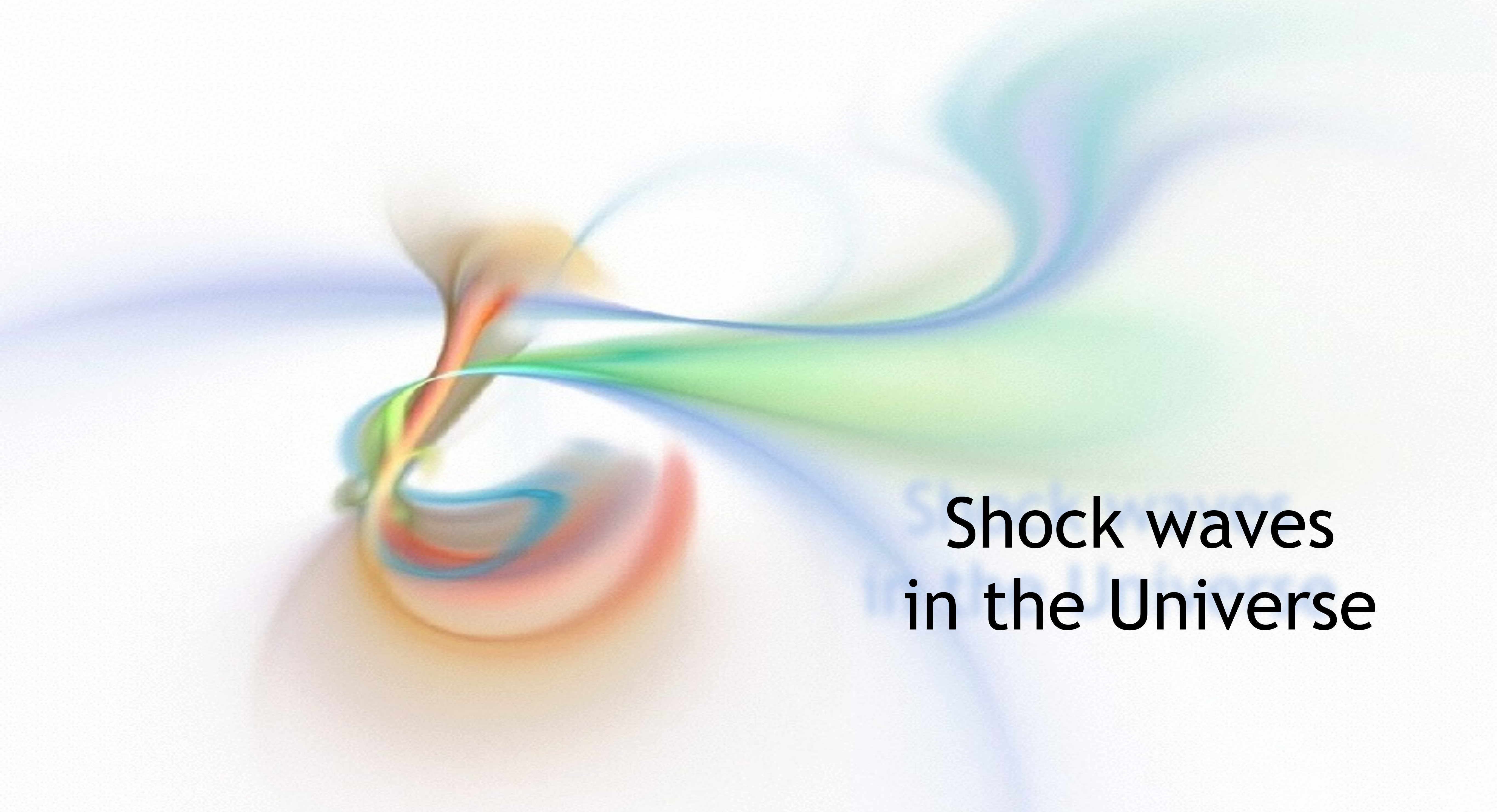
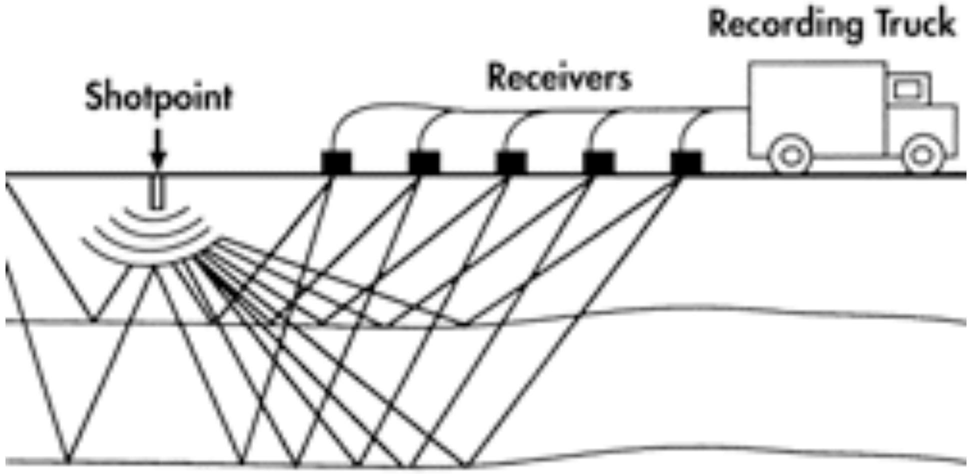


Shock waves in the Universe

Isabelle Grenier
AIM, Université Paris Diderot & CEA Saclay
Institut Universitaire de France

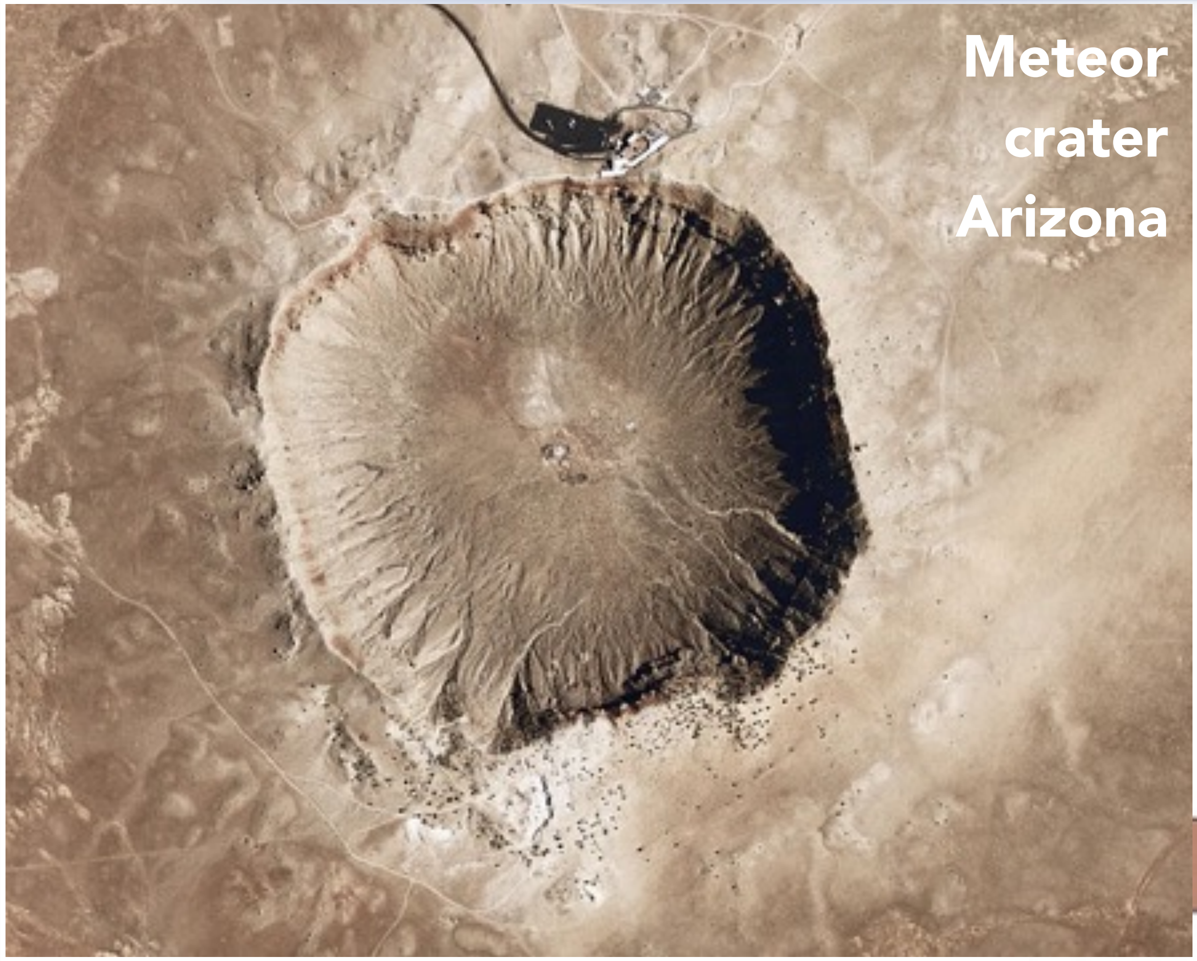
shock waves in geophysics

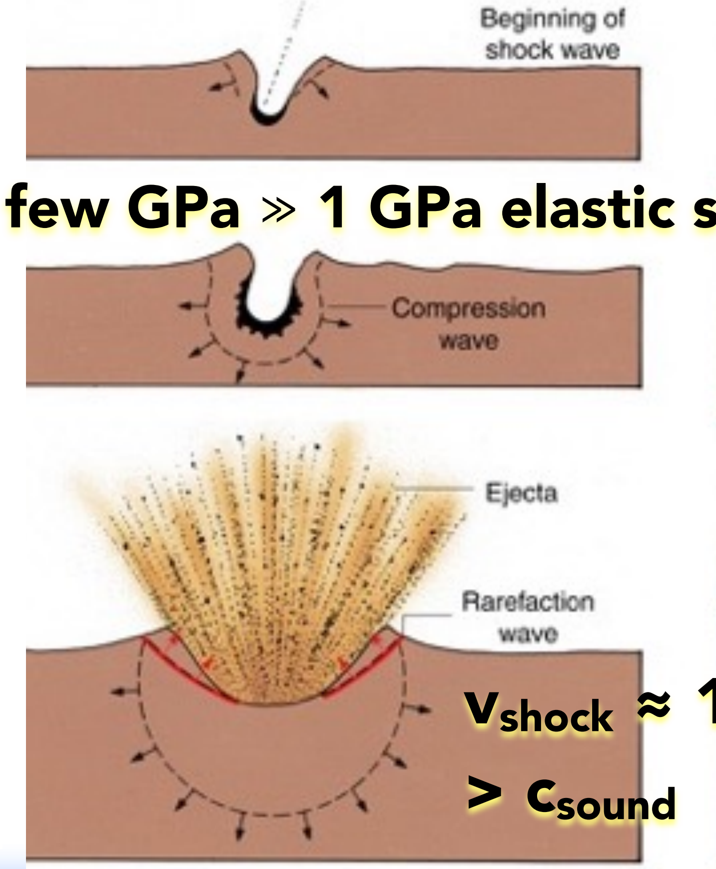
- lab exercise

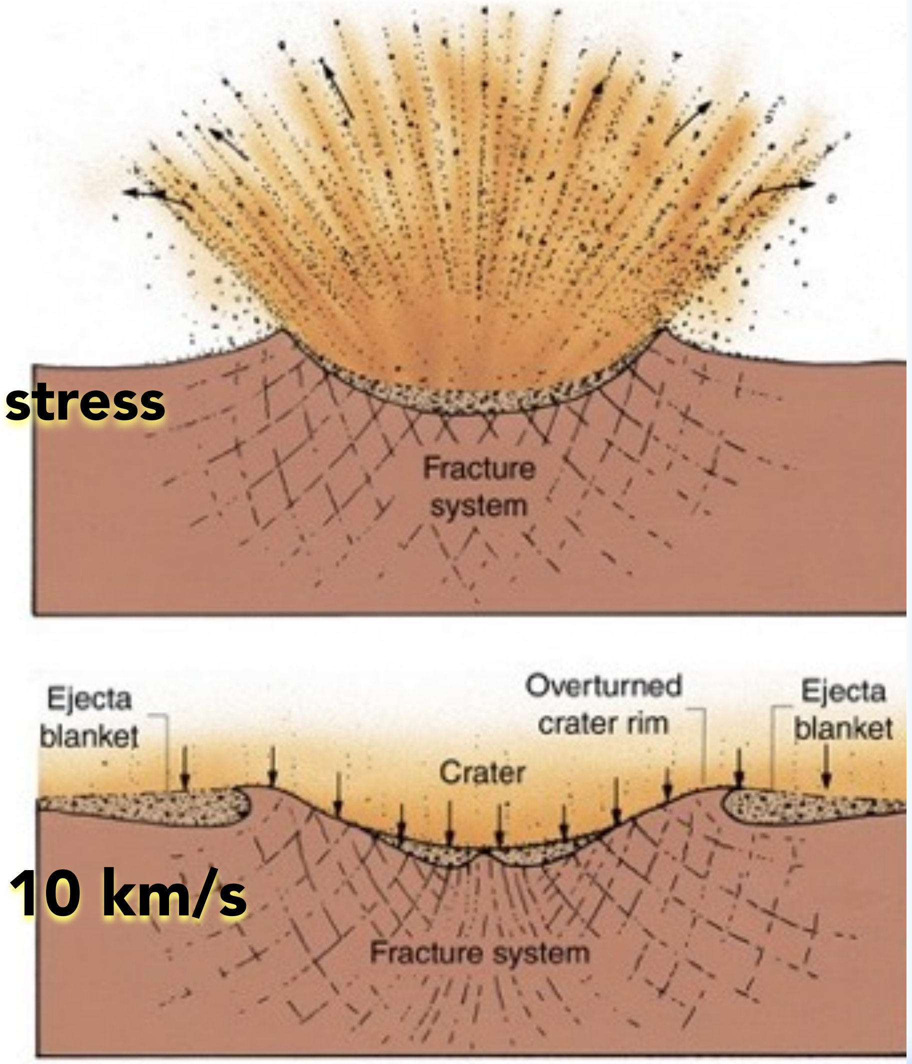


- professional exercise trucks pounding on the ground and recording the echos

why round craters from inclined impacts?



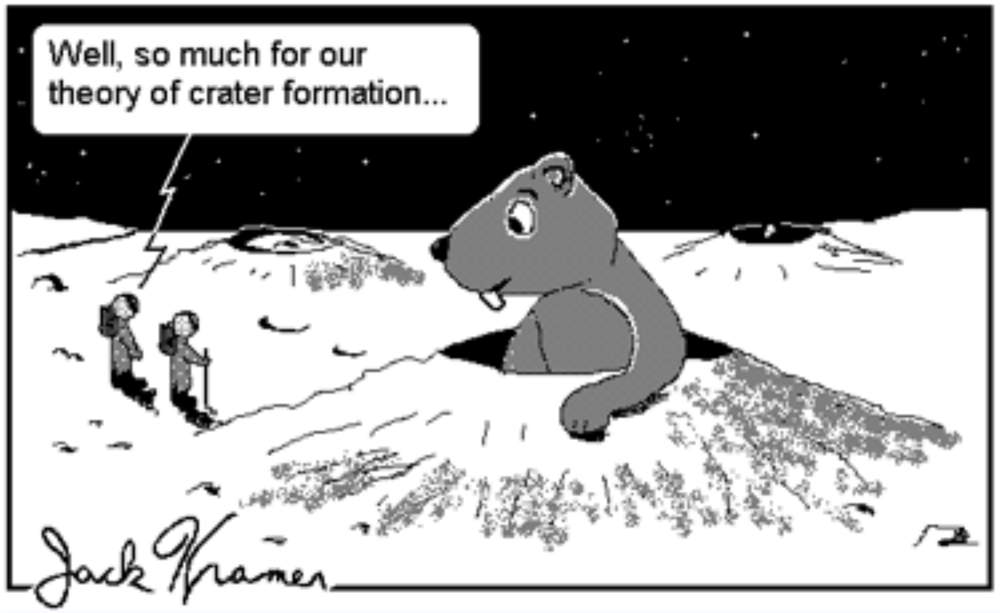




few GPa ≫ 1 GPa elastic stress

$v_{shock} \approx 10$ km/s

$> c_{sound}$



Pluto & Charon

- Pluto
- new horizons 2015

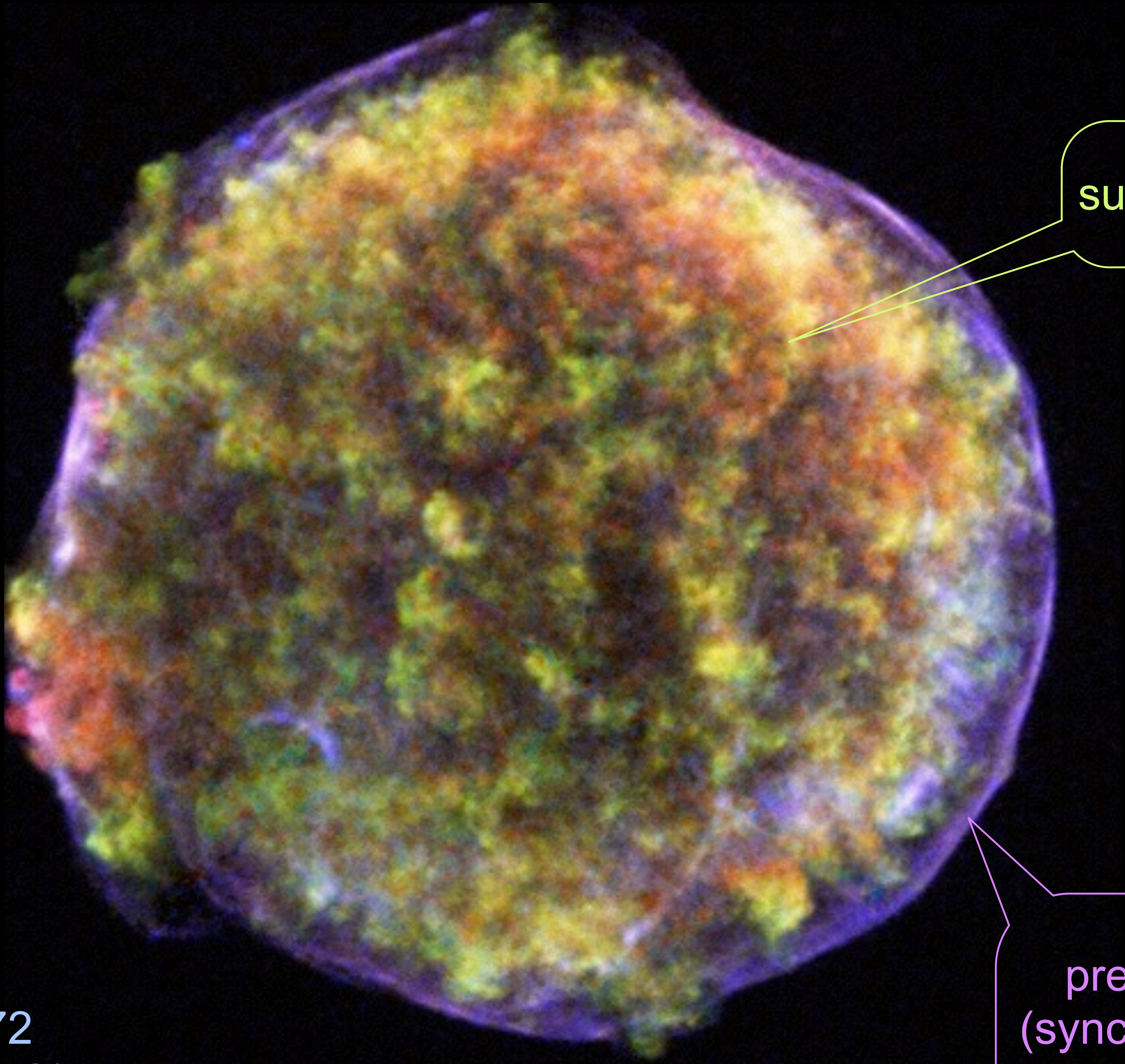
Charon

explosive shock waves

- Sarychev volcano in Russian Kuril island: plume, pyroclastic flow and shock wave
- seen from ISS june 12, 2009

explosive shock waves

- supernova explosion: ejecta at Mach M → (1-2) 10^4



supernova ejecta

preceding shock wave
(synchrotron radiation from
in-situ accelerated e^-)

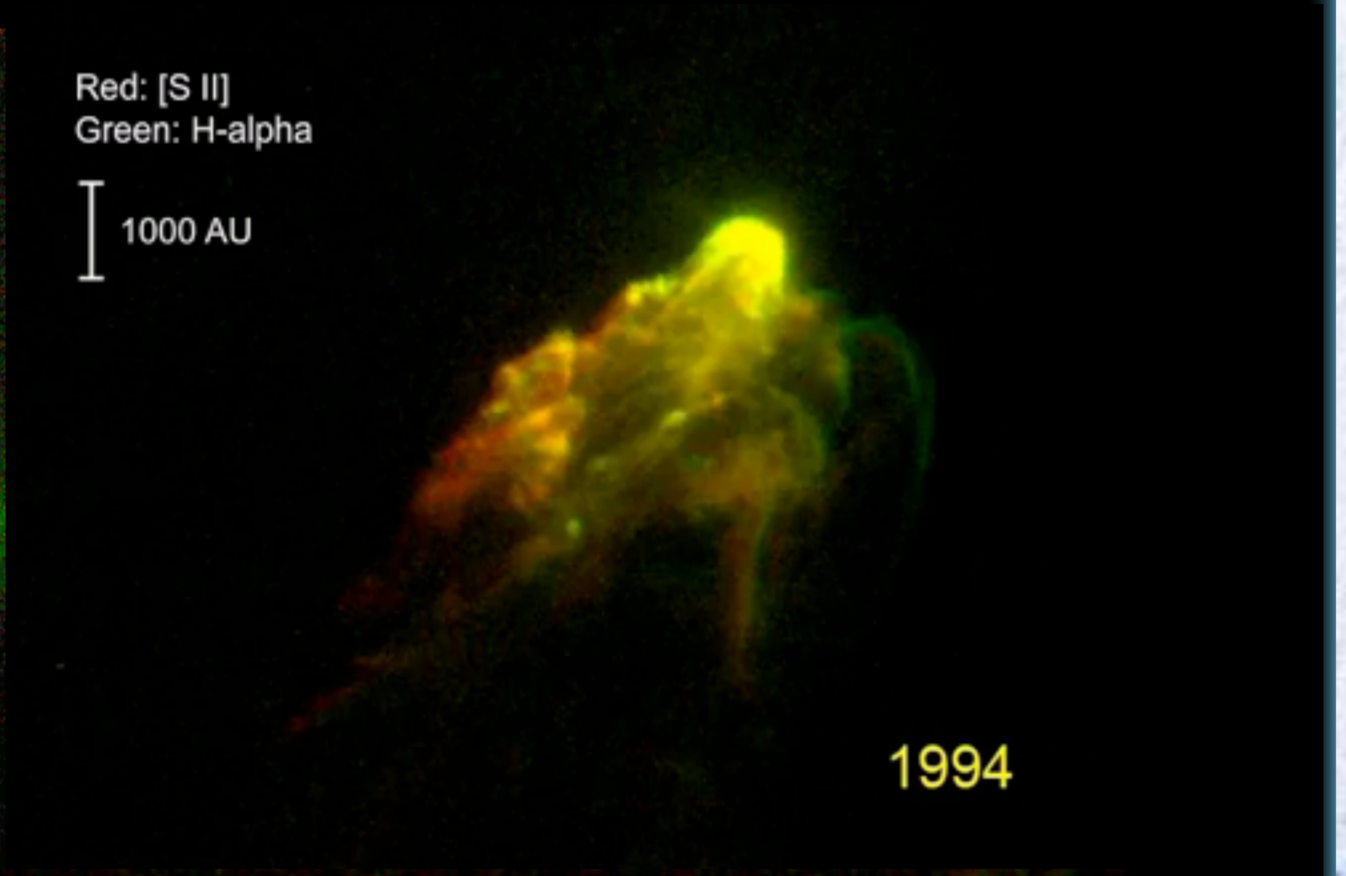
Tycho 1572
X rays (CXO)

beamed shock waves

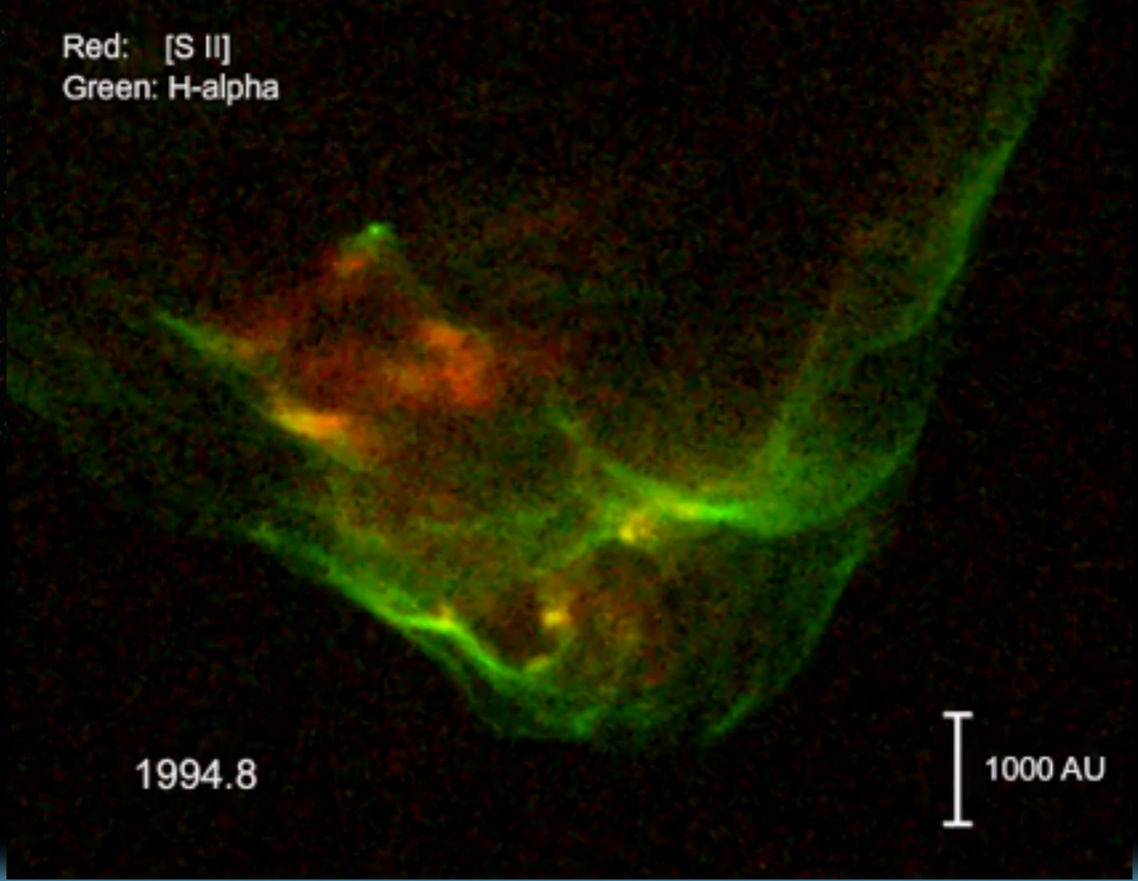
- supersonic stellar jets at Mach > 10

SII cooling

Hα hot (shock)



- bow shocks

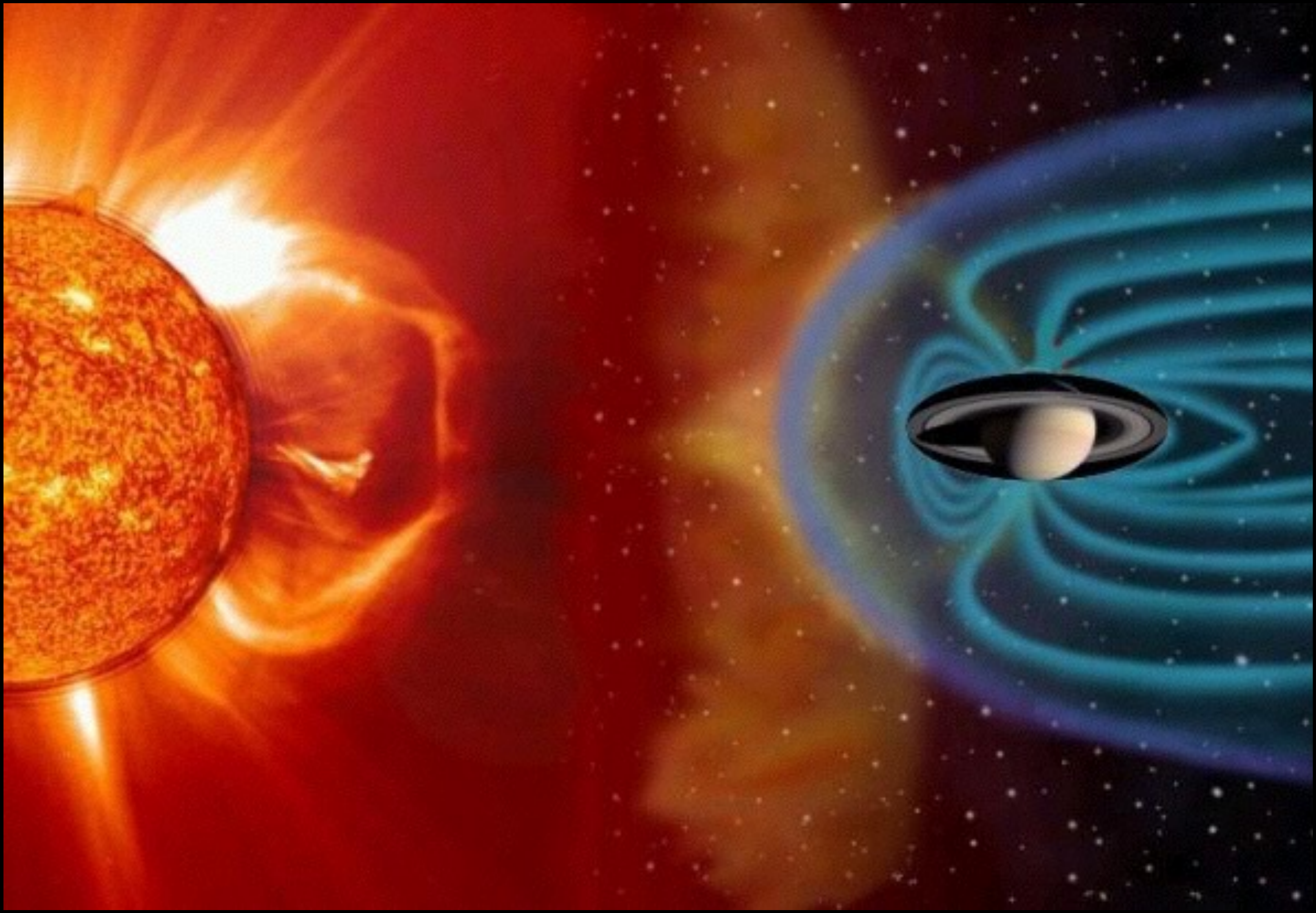


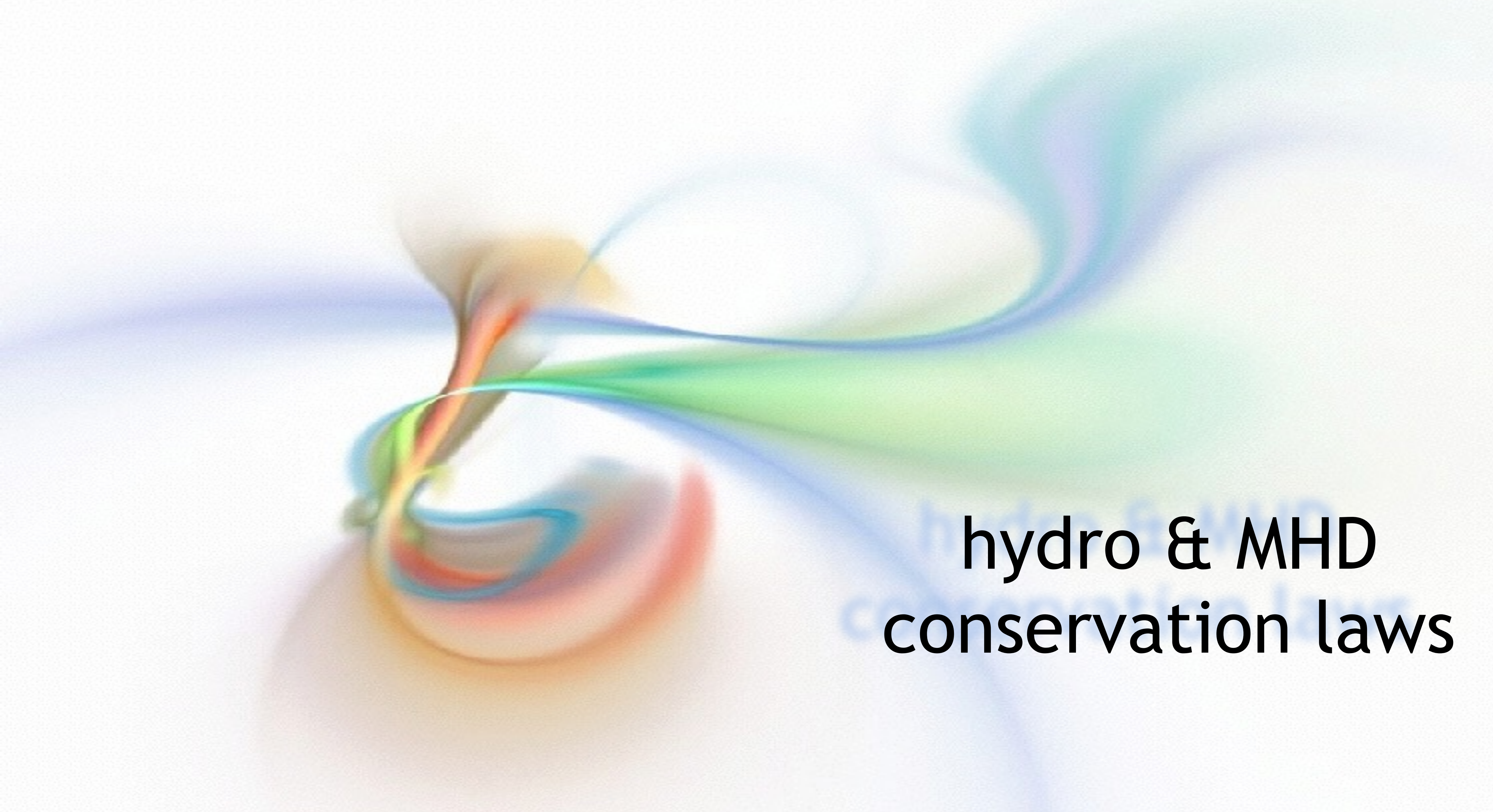
standing bow shocks

- star motion in the interstellar gas at Mach ~ 100

Bow Shock Around LL Orionis

magnetospheric shocks





hydro & MHD conservation laws

moving with the flow

- Euler vs. Lagrange
  - ✦ Euler: control volume focusing on a fixed region in the flowfield (instead of looking at the whole flowfield at once)
  - ✦ Lagrange: moving control volume following the same fluid elements
- time variation between $\rho(x_1,y_1,z_1,t_1)$ and $\rho(x_2,y_2,z_2,t_2)$
  - ✦ Taylor expansion:

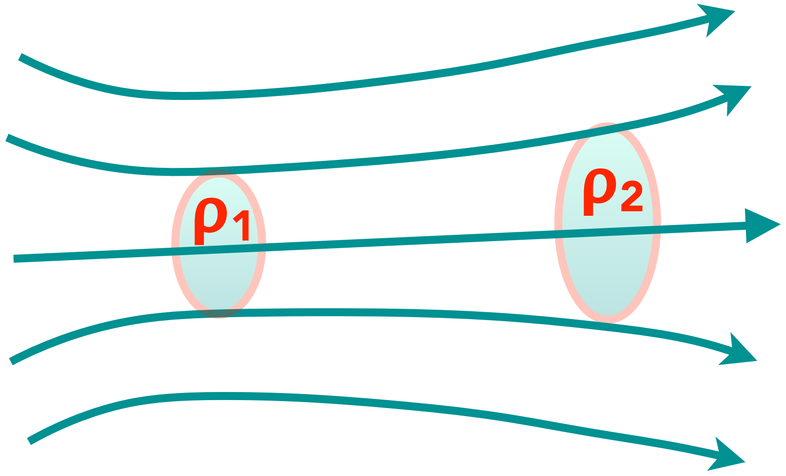
$$\lim_{t_2\to t_1}\frac{\rho_2-\rho_1}{t_2-t_1}=\frac{D\rho}{Dt}$$

$$\rho_2=\rho_1+\left(\frac{\partial\rho}{\partial x}\right)_1(x_2-x_1)+\left(\frac{\partial\rho}{\partial y}\right)_1(y_2-y_1)+\left(\frac{\partial\rho}{\partial z}\right)_1(z_2-z_1)+\left(\frac{\partial\rho}{\partial t}\right)_1(t_2-t_1)+higher-order-terms$$

$$\frac{\rho_2-\rho_1}{t_2-t_1}=\left(\frac{\partial\rho}{\partial x}\right)_1\frac{x_2-x_1}{t_2-t_1}+\left(\frac{\partial\rho}{\partial y}\right)_1\frac{y_2-y_1}{t_2-t_1}+\left(\frac{\partial\rho}{\partial z}\right)_1\frac{z_2-z_1}{t_2-t_1}+\left(\frac{\partial\rho}{\partial t}\right)_1$$

and $\lim_{t_2\to t_1}\frac{x_2-x_1}{t_2-t_1}=u$ $\quad\lim_{t_2\to t_1}\frac{y_2-y_1}{t_2-t_1}=v$ $\quad\lim_{t_2\to t_1}\frac{z_2-z_1}{t_2-t_1}=w$

  - ✦ thus the total derivative: $\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}$



$$\boxed{\lim_{t_2\to t_1}\frac{\rho_2-\rho_1}{t_2-t_1}=\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+\vec{v}\cdot\vec{\nabla}\rho}$$

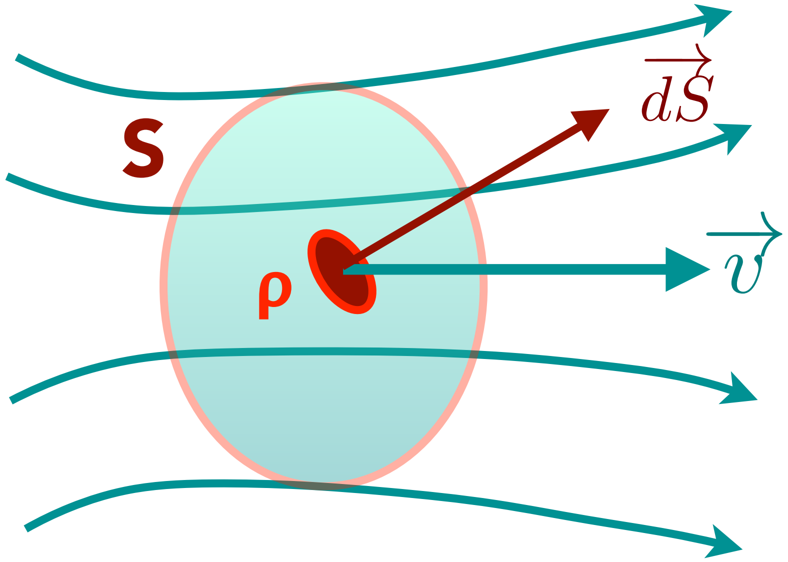
time variation	time variation due to
at fixed point	changes in flow properties
time derivative	convective derivative
Euler	Lagrange

mass conservation

- time variations of an α quantity = variations of α inside the volume + flux of α across surface

$$\oint_{\mathbf{S}} \vec{\alpha} \cdot \mathbf{d}\vec{S} = \oint_{\mathbf{V}} (\vec{\nabla} \cdot \vec{\alpha})\, \mathbf{dV}$$

$$\frac{\partial}{\partial \mathbf{t}}(\alpha\, \mathrm{density}) + \vec{\nabla} \cdot (\alpha\, \mathrm{flux}) = \mathrm{sources} - \mathrm{sinks}$$



- mass conservation

internal var. in/outflux

$$\frac{\partial}{\partial \mathbf{t}} \oint_{\mathbf{V}} \rho\, \mathbf{dV} = - \oint_{\mathbf{S}} \rho \vec{v} \cdot \mathbf{d}\vec{S} = \oint_{\mathbf{V}} \vec{\nabla} \cdot (\rho \vec{v})\, \mathbf{dV} \Rightarrow \oint_{\mathbf{V}} \left[\frac{\partial \rho}{\partial \mathbf{t}} + \vec{\nabla} \cdot (\rho \vec{v}) \right] \mathbf{dV} = \mathbf{0}$$

$$\vec{\nabla} \cdot (\rho \vec{v}) = \rho (\vec{\nabla} \cdot \vec{v}) + \vec{v} \cdot \vec{\nabla} \rho \quad \text{and} \quad \frac{\mathbf{D}\rho}{\mathbf{Dt}} = \frac{\partial \rho}{\partial \mathbf{t}} + \vec{v} \cdot \vec{\nabla} \rho$$

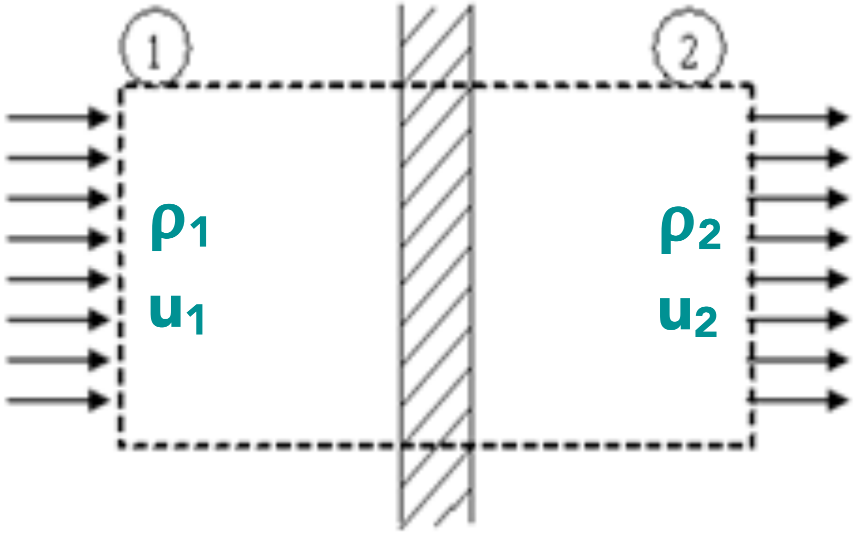
so mass conservation

$$\boxed{\frac{\partial \rho}{\partial \mathbf{t}} + \vec{\nabla} \cdot (\rho \vec{v}) = \mathbf{0} \quad \text{or} \quad \frac{\mathbf{D}\rho}{\mathbf{Dt}} = -\rho \vec{\nabla} \cdot \vec{v}}$$

total time variation due to velocity divergence in the flow

ex: diverging flow => ∂ρ/∂t < 0

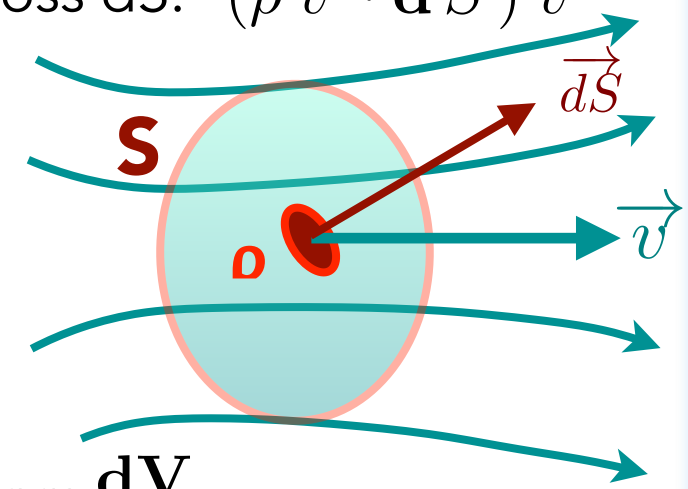
- case of a 1D steady flow: inward flux = outward flux



$$\boxed{\rho_1 \mathbf{u}_1 = \rho_2 \mathbf{u}_2}$$

momentum conservation

- Newton's law: fluid momentum in dV: $(\rho\mathbf{dV})\vec{v}$
  - ✦ mass flow across dS: $\rho\vec{v}\cdot\mathbf{d}\vec{S}$ => momentum flow per unit time across dS: $(\rho\vec{v}\cdot\mathbf{d}\vec{S})\vec{v}$
  - ✦ pressure force acting on dS from outside: $-\mathbf{p}\,d\vec{S}$
  - ✦ external force per unit mass from potential energy E_{pm} => force on mass in dV: $-(\rho\mathbf{dV})\vec{\nabla}\mathbf{E_{pm}}$



- Newton's law for a perfect fluid (no viscosity)

$$\frac{\mathbf{D}(\mathbf{m}\vec{v})}{\mathbf{Dt}} = \frac{\partial}{\partial\mathbf{t}}\oint_{\mathbf{V}}\rho\vec{v}\,\mathbf{dV} + \oint_{\mathbf{S}}(\rho\vec{v}\cdot d\vec{S})\vec{v} = -\oint_{\mathbf{S}}\mathbf{p}d\vec{S} - \oint_{\mathbf{V}}\rho\vec{\nabla}\mathbf{E_{pm}}\,\mathbf{dV}$$

internal var. in/outflux ext. pressure forces in volume

- ✦ using Ostrogradsky $\oint_{\mathbf{S}}\mathbf{p}\,d\vec{S} = \oint_{\mathbf{V}}(\vec{\nabla}\mathbf{p})\,\mathbf{dV}$ and $\frac{\mathbf{D}(\mathbf{m}\vec{v})}{\mathbf{Dt}} = \oint_{\mathbf{V}}\rho\left(\frac{\mathbf{D}\vec{v}}{\mathbf{Dt}}\right)\mathbf{dV}$ (next slide)
  one obtains per unit volume:

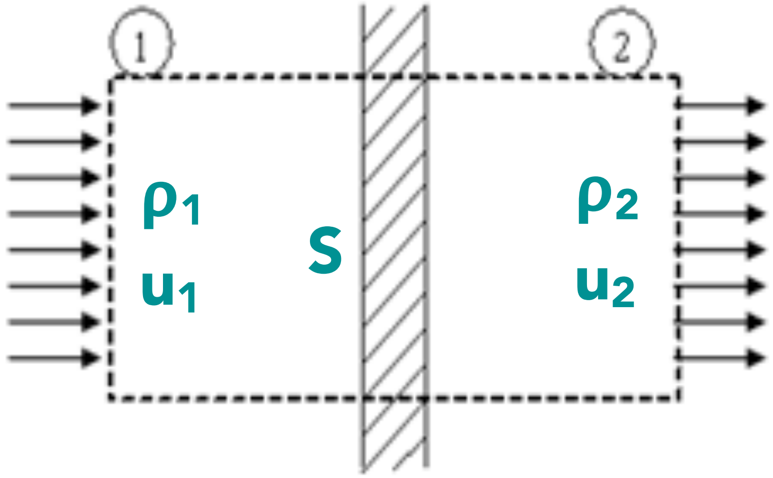
$$\rho\left[\frac{\partial\vec{v}}{\partial\mathbf{t}} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = \boxed{\rho\left(\frac{\mathbf{D}\vec{v}}{\mathbf{Dt}}\right) = -\vec{\nabla}\mathbf{p} - \rho\vec{\nabla}\mathbf{E_{pm}}} = \rho\left[\frac{\partial\vec{v}}{\partial\mathbf{t}} + \vec{\nabla}\left(\frac{\mathbf{v^2}}{\mathbf{2}}\right) + (\vec{\nabla}\wedge\vec{v})\wedge\vec{v}\right]$$

- case of a 1D steady flow without external forces:

$$\oint_{\mathbf{S}}(\rho\vec{v}\cdot d\vec{S})\vec{v} = -\oint_{\mathbf{S}}\mathbf{p}d\vec{S}$$

$$\Rightarrow \rho_1(-\mathbf{u_1}\mathbf{S})\mathbf{u_1} + \rho_2(\mathbf{u_2}\mathbf{S})\mathbf{u_2} = \mathbf{p_1}\mathbf{S} - \mathbf{p_2}\mathbf{S}$$

$$\boxed{\mathbf{p_1} + \rho_1\mathbf{u_1^2} = \mathbf{p_2} + \rho_2\mathbf{u_2^2}}$$



momentum time variation

- alternative form of the time variation in momentum:

$$\frac{\mathbf{D}(\mathbf{m}\vec{v})}{\mathbf{Dt}} = \frac{\partial}{\partial \mathbf{t}} \oint_{\mathbf{V}} \rho \vec{v}\, \mathbf{dV} + \oint_{\mathbf{S}} (\rho \vec{v} \cdot \vec{dS}) \vec{v}$$

- projected onto the x axis:

$$\frac{\partial}{\partial \mathbf{t}} \oint_{\mathbf{V}} \rho \mathbf{v_x}\, \mathbf{dV} + \oint_{\mathbf{S}} \mathbf{v_x} (\rho \vec{v} \cdot \vec{dS}) = \frac{\partial}{\partial \mathbf{t}} \oint_{\mathbf{V}} \rho \mathbf{v_x}\, \mathbf{dV} + \oint_{\mathbf{V}} \vec{\nabla} \cdot (\rho \mathbf{v_x} \vec{v})\, \mathbf{dV} = \oint_{\mathbf{V}} \left[\frac{\partial \rho \mathbf{v_x}}{\partial \mathbf{t}} + \vec{\nabla} \cdot (\rho \mathbf{v_x} \vec{v}) \right] \mathbf{dV}$$

- using $\vec{\nabla} \cdot (\rho \mathbf{v_x} \vec{v}) = \vec{\nabla} \cdot [\mathbf{v_x}(\rho \vec{v})] = \mathbf{v_x} \vec{\nabla} \cdot (\rho \vec{v}) + \rho \vec{v} \cdot \vec{\nabla} \mathbf{v_x}$

  and $\dfrac{\partial (\rho \mathbf{v_x})}{\partial \mathbf{t}} = \rho \dfrac{\partial \mathbf{v_x}}{\partial \mathbf{t}} + \mathbf{v_x} \dfrac{\partial \rho}{\partial \mathbf{t}}$

- and regrouping terms in ρ and in v_x, the integrand becomes

$$\rho \left[\underbrace{\frac{\partial \mathbf{v_x}}{\partial \mathbf{t}} + \vec{v} \cdot \vec{\nabla} \mathbf{v_x}}_{\frac{\mathbf{Dv_x}}{\mathbf{Dt}}} \right] + \mathbf{v_x} \left[\cancel{\frac{\partial \rho}{\partial \mathbf{t}} + \vec{\nabla} \cdot (\rho \vec{v})} \right]$$

0 from mass conservation

- same on other axes, therefore:

$$\frac{\mathbf{D}(\mathbf{m}\vec{v})}{\mathbf{Dt}} = \oint_{\mathbf{V}} \rho \left(\frac{\mathbf{D}\vec{v}}{\mathbf{Dt}} \right) \mathbf{dV}$$

energy conservation

- power conservation from 1st law of thermodynamics $\mathbf{\frac{dE}{dt} = \frac{\delta Q}{\delta t} + \frac{\delta W}{\delta t}}$
  - ✦ e_{int} = internal energy density
  - ✦ total kinetic energy per unit volume $e_{kin} = u_{int} + \rho v^2/2$
  - ✦ net heat rate per unit volume (radiative+conductive): dQ/dVdt = Γ (gain) - Λ (loss)
  - ✦ pressure work dW = -pdV = -p dS v dt => flux dW/dSdt = -pv

internal var. in/outflux pressure work mech work heat change

$$\frac{\partial}{\partial \mathbf{t}} \oint_{\mathbf{V}} \rho \left(\mathbf{e_{int}} + \frac{\mathbf{v^2}}{\mathbf{2}}\right) \mathbf{dV} + \oint_{\mathbf{S}} \rho \left(\mathbf{e_{int}} + \frac{\mathbf{v^2}}{\mathbf{2}}\right) \vec{v} \cdot \vec{dS} = -\oint_{\mathbf{S}} \mathbf{p}\vec{dS} \cdot \vec{v} - \oint_{\mathbf{V}} (\rho\, \mathbf{dV}\, \vec{\nabla} \mathbf{E_{pm}}) \cdot \vec{v} + \oint_{\mathbf{V}} (\mathbf{\Gamma} - \mathbf{\Lambda})\, \mathbf{dV}$$

using the divergence theorem to surface integrals and equating the volume integrands

$$\frac{\partial}{\partial \mathbf{t}}(\rho\, \mathbf{e_{kin}}) + \vec{\nabla} \cdot (\mathbf{e_{kin}} \rho \vec{v}) = -\vec{\nabla} \cdot (\mathbf{p} \vec{v}) - \rho \vec{\nabla} \mathbf{E_{pm}} \cdot \vec{v} + (\mathbf{\Gamma} - \mathbf{\Lambda})$$

using $\frac{\partial}{\partial \mathbf{t}}(\rho\, \mathbf{e_{kin}}) = \rho \frac{\partial \mathbf{e_{kin}}}{\partial \mathbf{t}} + \mathbf{e_{kin}} \frac{\partial \rho}{\partial \mathbf{t}}$ and $\vec{\nabla} \cdot (\mathbf{e_{kin}} \rho \vec{v}) = \mathbf{e_{kin}} \vec{\nabla} \cdot (\rho \vec{v}) + (\rho \vec{v}) \cdot \vec{\nabla} \mathbf{e_{kin}}$

the left hand side term becomes $\rho \left[\frac{\partial \mathbf{e_{kin}}}{\partial \mathbf{t}} + \vec{v} \cdot \vec{\nabla} \mathbf{e_{kin}}\right] + \mathbf{e_{kin}} \left[\cancel{\frac{\partial \rho}{\partial \mathbf{t}} + \vec{\nabla} \cdot (\rho \vec{v})}\right] = \rho \frac{\mathbf{D e_{kin}}}{\mathbf{Dt}}$

so the power conservation yields

$$\boxed{\rho \frac{\mathbf{D}}{\mathbf{Dt}} \left[\mathbf{e_{int}} + \frac{\mathbf{v^2}}{\mathbf{2}}\right] = -\vec{\nabla} \cdot (\mathbf{p} \vec{v}) - \rho \vec{\nabla} \mathbf{E_{pm}} \cdot \vec{v} + (\mathbf{\Gamma} - \mathbf{\Lambda})}$$

energy conservation: alternative forms

- development of the mechanical work:

$$\vec{\nabla}\cdot(\rho\mathbf{E_{pm}}\vec{v}) = \mathbf{E_{pm}}\left[\vec{\nabla}\cdot(\rho\vec{v})\right] + \rho\vec{v}\cdot\vec{\nabla}\mathbf{E_{pm}}$$

$$\Rightarrow \mathbf{W_{mech}} = -\rho\vec{v}\cdot\vec{\nabla}\mathbf{E_{pm}} = -\vec{\nabla}\cdot(\rho\mathbf{E_{pm}}\vec{v}) + \mathbf{E_{pm}}\left[\vec{\nabla}\cdot(\rho\vec{v})\right]$$

- mass conservation =>

$$\mathbf{W_m} = -\vec{\nabla}\cdot(\mathbf{E_{pm}}\rho\vec{v}) - \mathbf{E_{pm}}\frac{\partial\rho}{\partial \mathbf{t}} = -\vec{\nabla}\cdot(\mathbf{E_{pm}}\rho\vec{v}) - \frac{\partial(\rho\mathbf{E_{pm}})}{\partial \mathbf{t}} + \rho\frac{\partial\mathbf{E_{pm}}}{\partial \mathbf{t}}$$

- power conservation

$$\frac{\partial}{\partial \mathbf{t}}(\rho\,\mathbf{e_{kin}}) + \vec{\nabla}\cdot(\mathbf{e_{kin}}\rho\vec{v}) = -\vec{\nabla}\cdot(\mathbf{p}\,\vec{v}) - \vec{\nabla}\cdot(\mathbf{E_{pm}}\rho\vec{v}) - \frac{\partial(\rho\mathbf{E_{pm}})}{\partial \mathbf{t}} + \rho\frac{\partial\mathbf{E_{pm}}}{\partial \mathbf{t}} + (\mathbf{\Gamma}-\mathbf{\Lambda})$$

- regrouping

$$\boxed{\frac{\partial}{\partial \mathbf{t}}\left[\rho\mathbf{e_{int}} + \rho\frac{\mathbf{v^2}}{\mathbf{2}} + \rho\mathbf{E_{pm}}\right] + \vec{\nabla}\cdot\left(\rho\mathbf{e_{int}} + \rho\frac{\mathbf{v^2}}{\mathbf{2}} + \rho\mathbf{E_{pm}} + \mathbf{p}\right)\vec{v} = \rho\frac{\partial\mathbf{E_{pm}}}{\partial \mathbf{t}} + (\mathbf{\Gamma}-\mathbf{\Lambda})}$$

- case of a 1D steady flow without external force:

$$\rho_1(\mathbf{e_{int1}} + \frac{\mathbf{u_1^2}}{\mathbf{2}})(-\mathbf{u_1S}) + \rho_2(\mathbf{e_{int2}} + \frac{\mathbf{u_2^2}}{\mathbf{2}})(\mathbf{u_2S}) = -\mathbf{p_1}(-\mathbf{u_1S}) - \mathbf{p_2u_2S} + \mathbf{Q_{net}}$$

$$\boxed{\Rightarrow \left[\rho_2\mathbf{e_{int2}} + \rho_2\frac{\mathbf{u_2^2}}{\mathbf{2}} + \mathbf{p_2}\right]\mathbf{u_2} = \left[\rho_1\mathbf{e_{int1}} + \rho_1\frac{\mathbf{u_1^2}}{\mathbf{2}} + \mathbf{p_1}\right]\mathbf{u_1} + \frac{\mathbf{Q_{net}}}{\mathbf{S}}}$$

équation of state

- adiabatic equation of state to close the system of equations:

$$\boxed{p = K\rho^{\gamma}}$$

$\gamma = C_p/C_V = 5/3$ (ideal or monoatomic gas),

$\gamma = 7/5$ (diatomic gas),

$\gamma = 4/3$ (relativistic gas)

isothermal transformation $\gamma = 1$

- if ideal gas: internal energy $U_{int} = C_V T$ and gas law $pV = \nu RT$ with $C_p - C_V = \nu R$

therefore

$$\mathbf{U_{int}} = \frac{\nu \mathbf{R}}{\gamma - \mathbf{1}}\mathbf{T} = \frac{\mathbf{pV}}{\gamma - \mathbf{1}} \qquad \text{and enthalpy} \quad \mathbf{H} = \mathbf{U_{int}} + \mathbf{pV}$$

enthalpy

$$\boxed{\mathbf{e_{int}} = \frac{\mathbf{U_{int}}}{\mathbf{V}} = \frac{\mathbf{p}}{\gamma - \mathbf{1}} \qquad \text{and} \quad \mathbf{h} = \mathbf{u_{int}} + \mathbf{p} = \frac{\gamma \mathbf{p}}{\gamma - \mathbf{1}}}$$

MHD

- Maxwell's equations with slow variations
  $v_e \approx v_{ion}$ since very small $|v_e - v_{ion}|$ necessary
  to generate B

$$\vec{\nabla} \wedge \vec{B} = \mu \vec{j} \left(+\frac{\partial \vec{D}}{\partial \mathbf{t}} \ll\right) \qquad \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial \mathbf{t}}$$

$$\vec{\nabla} \cdot \vec{B} = \mathbf{0} \qquad \vec{\nabla} \cdot \vec{E} = -\frac{\rho_{\mathbf{c}}}{\epsilon}$$

- Ohm's law in the plasma rest frame (*) and
  Lorentz transformation ($\gamma \approx 1$) of E to the observer frame
  j = speed difference = Lorentz invariant

$$\vec{j}^{\,*} = \sigma \vec{E}^{*} \quad \text{et} \quad \vec{j}^{\,*} = \vec{j}$$

$$\vec{E}^{*} = \gamma(\vec{E} + \vec{v} \wedge \vec{B}) \approx \vec{E} + \vec{v} \wedge \vec{B} \Rightarrow \vec{j} = \sigma\left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$

- induction equation:

$$\vec{\nabla} \wedge \vec{B} = \mu \vec{j} \text{ et } \vec{j} = \sigma\left(\vec{E} + \vec{v} \wedge \vec{B}\right) \Rightarrow \vec{E} = -\vec{v} \wedge \vec{B} + \frac{\vec{\nabla} \wedge \vec{B}}{\mu\sigma}$$

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \wedge \vec{E} = \vec{\nabla} \wedge \left(\vec{v} \wedge \vec{B}\right) - \eta \vec{\nabla} \wedge \left(\vec{\nabla} \wedge \vec{B}\right) = \vec{\nabla} \wedge \left(\vec{v} \wedge \vec{B}\right) - \eta \vec{\nabla}(\cancel{\vec{\nabla} \cdot} \vec{B}) + \eta \nabla^2 B$$

$$\Rightarrow \frac{\partial \vec{B}}{\partial \mathbf{t}} = \vec{\nabla} \wedge (\vec{v} \wedge \vec{B}) + \eta \nabla^{\mathbf{2}} \mathbf{B}$$

$$\eta = \frac{\mathbf{1}}{\mu\sigma} = \text{magnetic diffusivity}$$

$$\frac{\mathbf{term2}}{\mathbf{term1}} \approx \frac{\mathbf{vL}}{\eta} = \text{magn. Reynolds nb}$$

✦ si R_m << 1 induction equation = diffusion equation
B inhomogeneities diffuse and fade out with speed $v_d \sim \eta/L$

ideal MHD

- ideal plasma: conductivity $\sigma \to \infty$ ⇒ $\vec{E} \to -\vec{v} \wedge \vec{B}$ and $\dfrac{\partial \vec{B}}{\partial t} \to \vec{\nabla} \wedge (\vec{v} \wedge \vec{B})$

  thus $R_m >> 1$, no diffusion

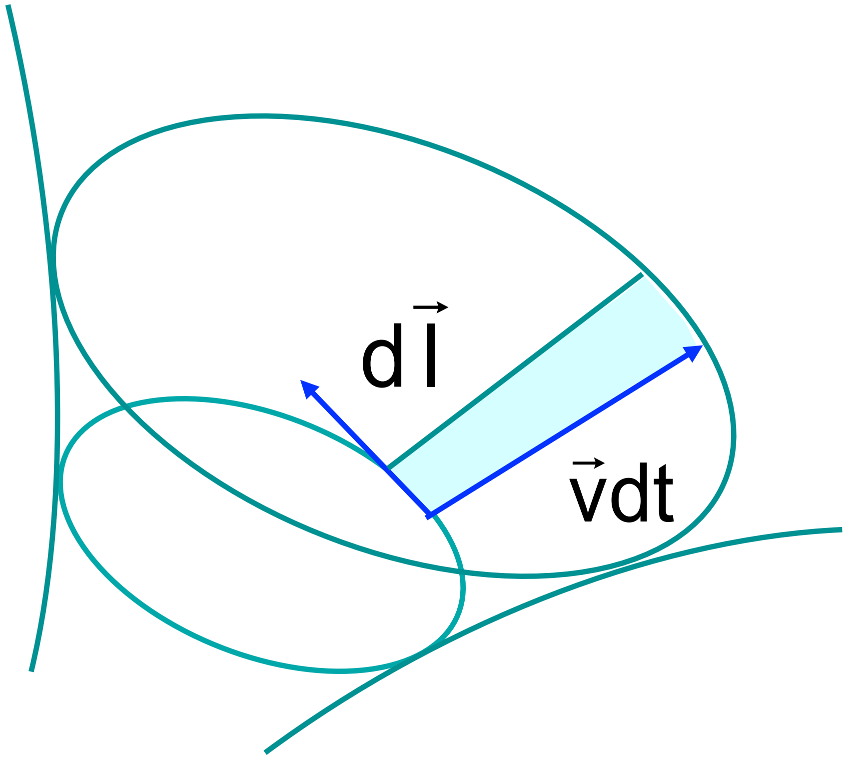
- B field lines frozen in the plasma (the flow carries them away)

  magnetic flux: time variations of B inside the tube + in/outflux of B across the tube walls

$$\frac{D}{Dt}\left[\int_S \vec{B}.d\vec{S}\right] = \int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S} + \int_C \vec{B}.(\vec{v} \wedge d\vec{\ell}) = \int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S} - \int_C (\vec{v} \wedge \vec{B}).d\vec{\ell}$$

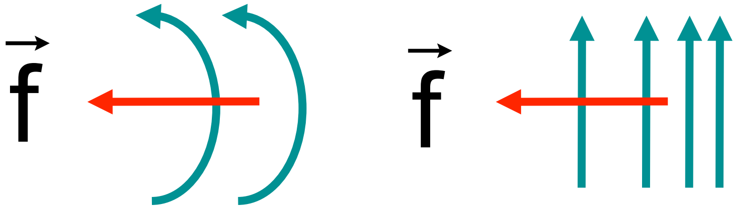
$$= \int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S} - \int_S \vec{\nabla} \wedge (\vec{v} \wedge \vec{B}).d\vec{S} = 0$$

car $$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \wedge (\vec{v} \wedge \vec{B})$$



- Lorentz force per unit volume:

$$\left.\begin{array}{l} \dfrac{dF_{Lorentz}}{dV} = \vec{j} \wedge \vec{B} \\ \\ \text{et } \vec{\nabla} \wedge \vec{B} = \mu \vec{j} \end{array}\right\} = \frac{(\vec{\nabla} \wedge \vec{B}) \wedge \vec{B}}{\mu} = \underbrace{(\vec{B}.\vec{\nabla})\frac{\vec{B}}{\mu}}_{\text{tension mgn.}} - \underbrace{\vec{\nabla}\left(\frac{B^2}{2\mu}\right)}_{\text{grad}(p_B)}$$



MHD conservations

- mass

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.(\rho\vec{v}) = 0$$

- momentum (+ B pressure + B tension)

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_k}\left[\rho v_i v_k + (p + \frac{B^2}{2\mu_0})\delta_{ik} - \frac{1}{\mu_0}B_i B_k\right] = -\rho\frac{dE_{pot}}{dx_i}$$

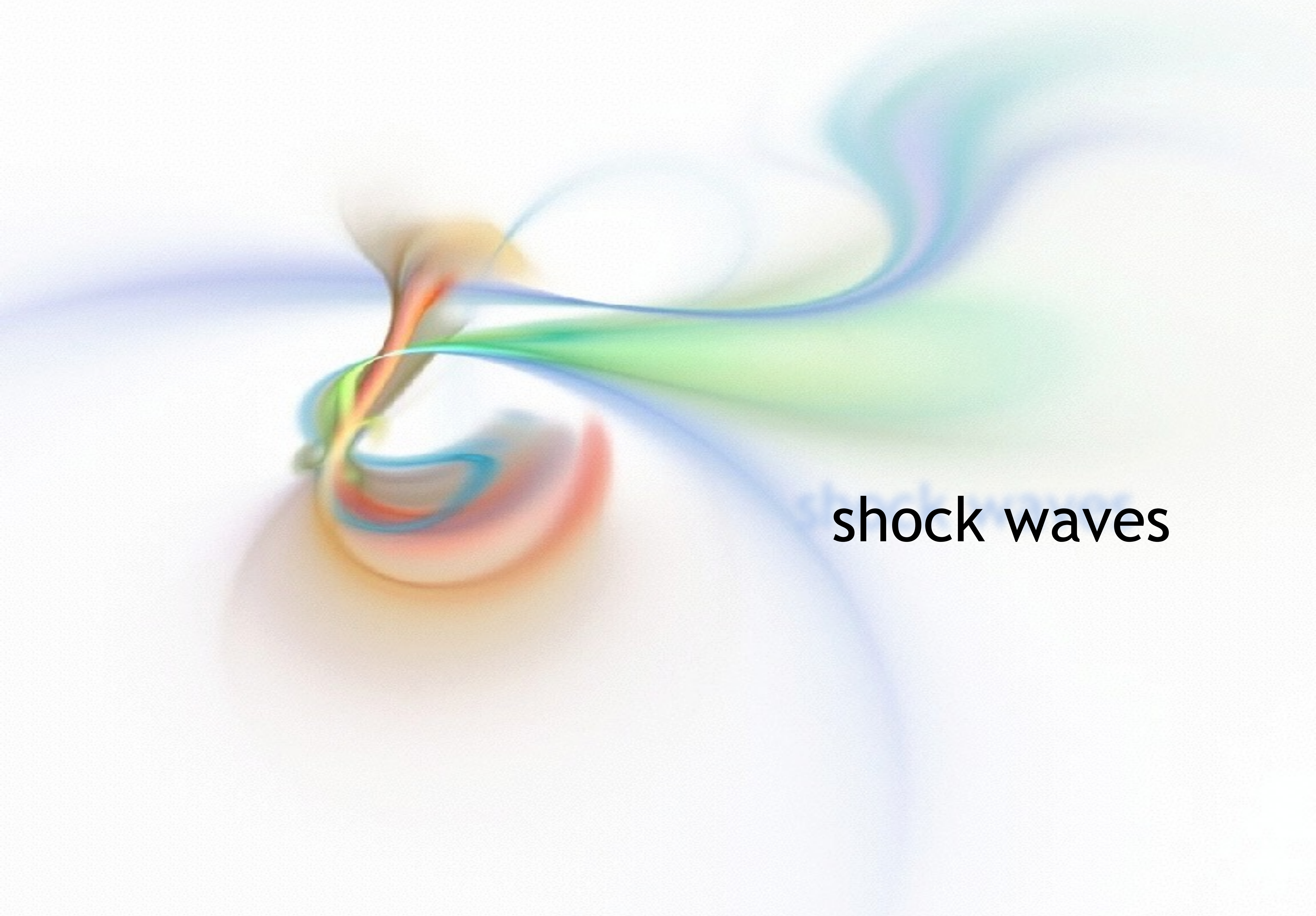
- energy (+ B energy density + Poynting flux with frozen B) $\vec{E} = -\vec{v} \wedge \vec{B}$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + u_{int} + \frac{B^2}{2\mu_0}\right] + \vec{\nabla}.\left[(\frac{1}{2}\rho v^2 + u_{int} + p)\vec{v} + \frac{(-\vec{v} \wedge \vec{B}) \wedge \vec{B}}{\mu_0}\right] = -\rho(\vec{\nabla}E_{pot}).\vec{v} + q$$

- Maxwell with ideal plasma

$$\frac{\partial\vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \frac{\partial\vec{B}}{\partial t} - \vec{\nabla} \wedge (\vec{v} \wedge \vec{B}) = 0$$

$$\vec{\nabla}.\vec{B} = 0$$



shock waves

supersonic motion

- sound speed

$$\text{adiabatic :} \quad \mathbf{c_s} = \sqrt{\gamma \left(\frac{\mathbf{p_0}}{\rho_\mathbf{0}} \right)}$$

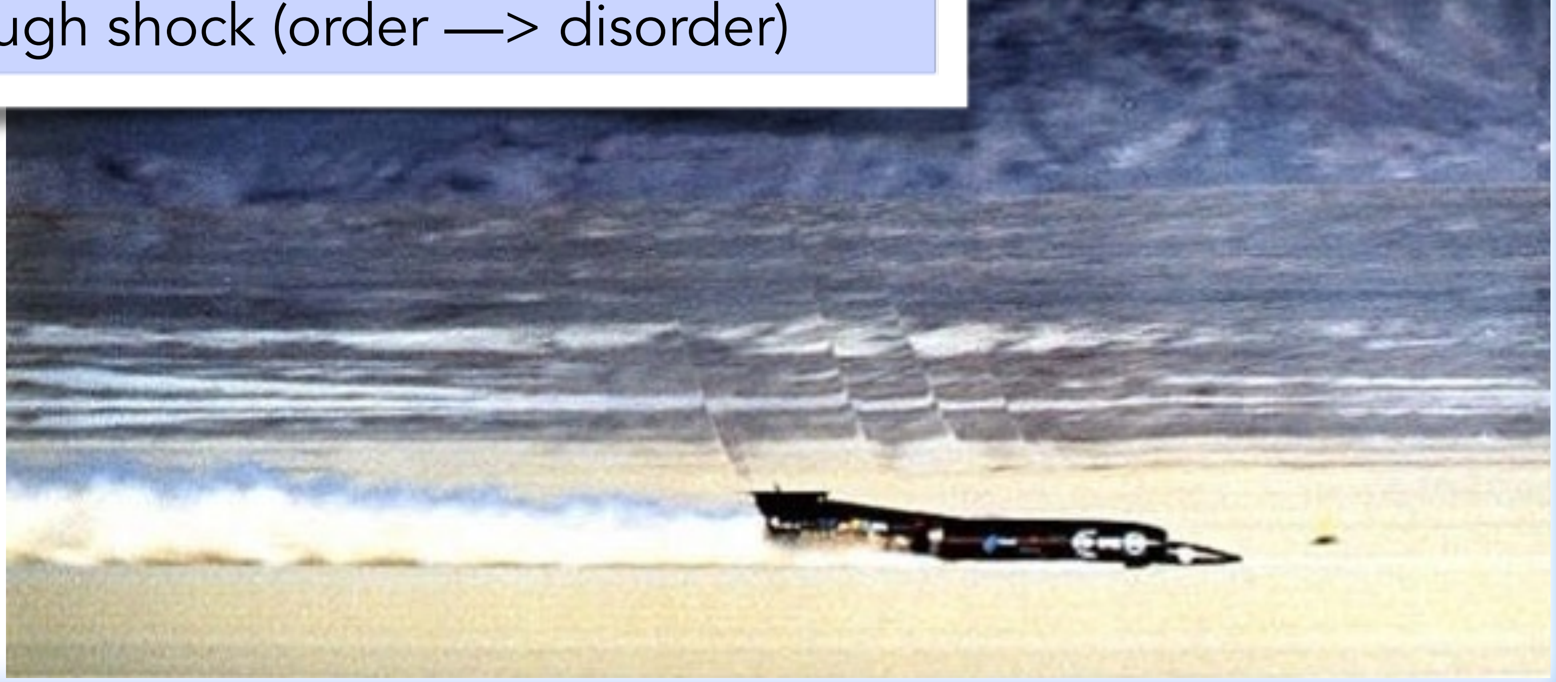
- ratio of macro vs. micro kinetic energy scales with M^2:

$$\frac{\frac{1}{2}\rho \mathbf{v^2}}{\mathbf{e_{int}}} = \frac{\gamma - \mathbf{1}}{\mathbf{2}} \mathbf{M^2}$$

- shockwave = discontinuity in ρ, p, T, and velocity

shock: convert organised $E_{kin} \approx \rho v^2/2$
to disorganised internal energy (e_{int}, T)
adiabatic shock, but entropy increase
through shock (order —> disorder)

- sub-sonic flow:
  - ✦ disturbance can be felt by the whole fluid domain.
- supersonic flow
  - ✦ no possible warning of the arrival of a disturbance



shock wave = discontinuity

- steepening of wave front
  ex: sound wave
  ex: incompressible MHD wave v(B)
  ex: compressible MHD wave B $\propto$ ρ (frozen)

$$c_s = \sqrt{\gamma \frac{p}{\rho}} \text{ et } p \propto \rho^{\gamma} \Rightarrow c_s \propto \rho^{\frac{\gamma-1}{2}}$$

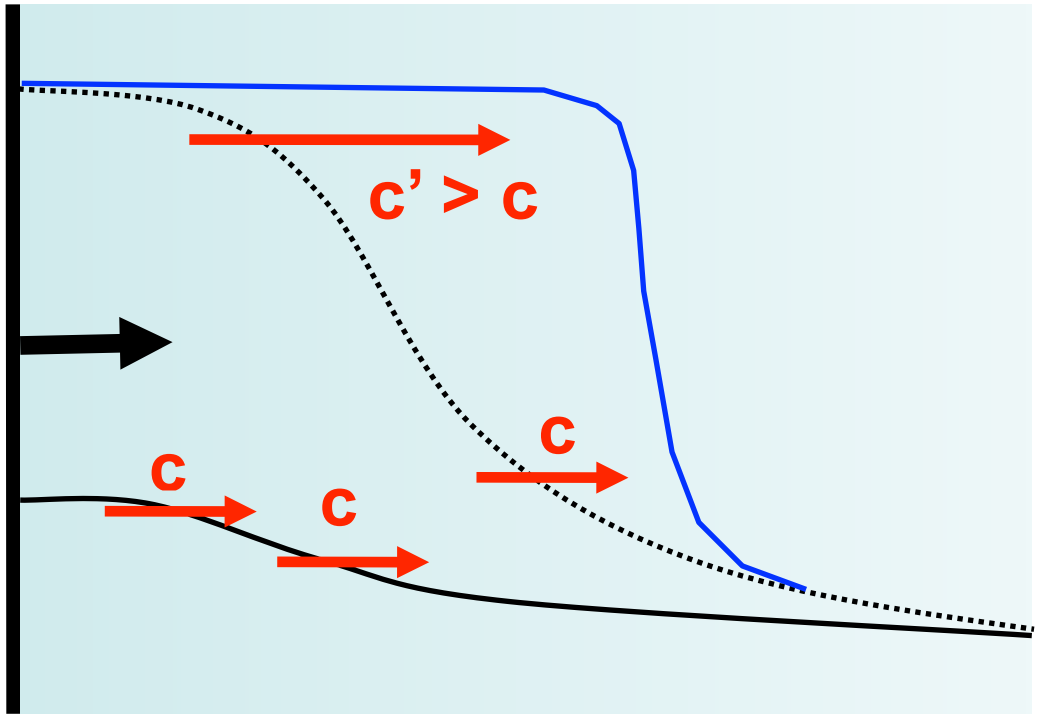
$$v_A = B/\sqrt{\mu_0\rho}$$

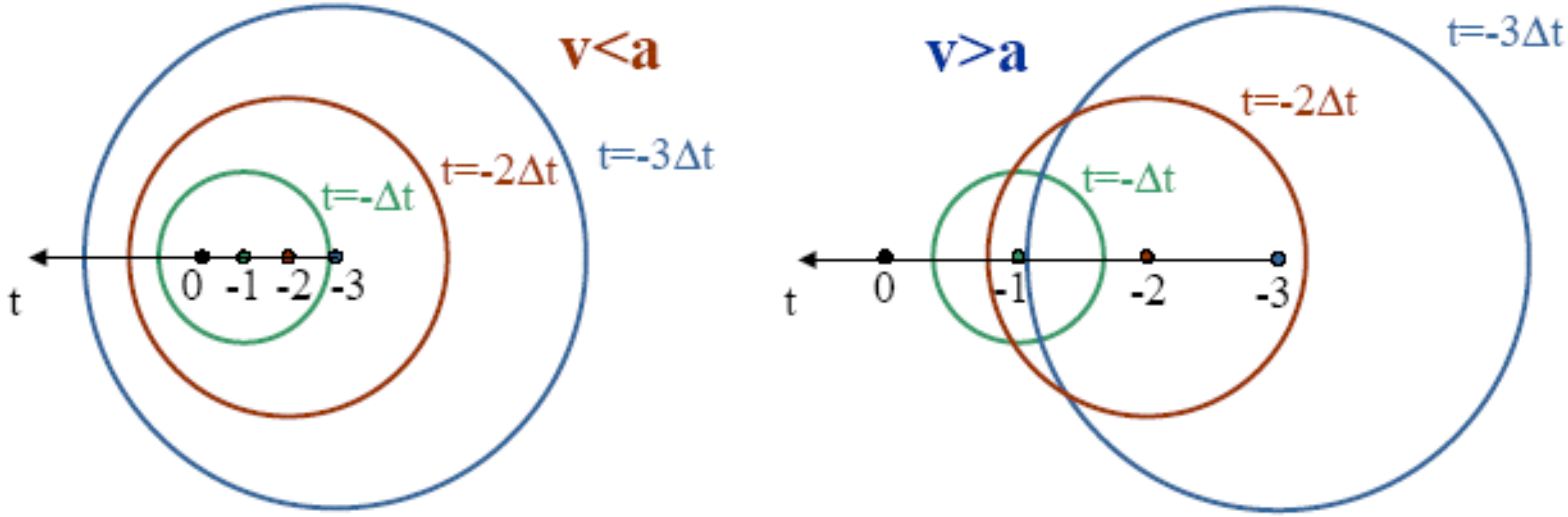
- supersonic boum
  - ✦ sound waves accumulate along a cone behind the source
  - ✦ no pressure information upstream

piston

pressure

profile

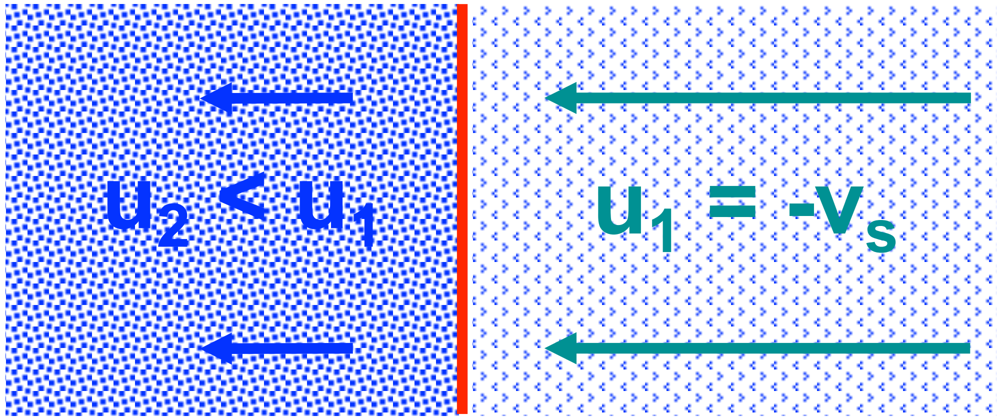




pressure discontinuity

- Papua New Guinea Tavurvur Volcano, August 29, 2014

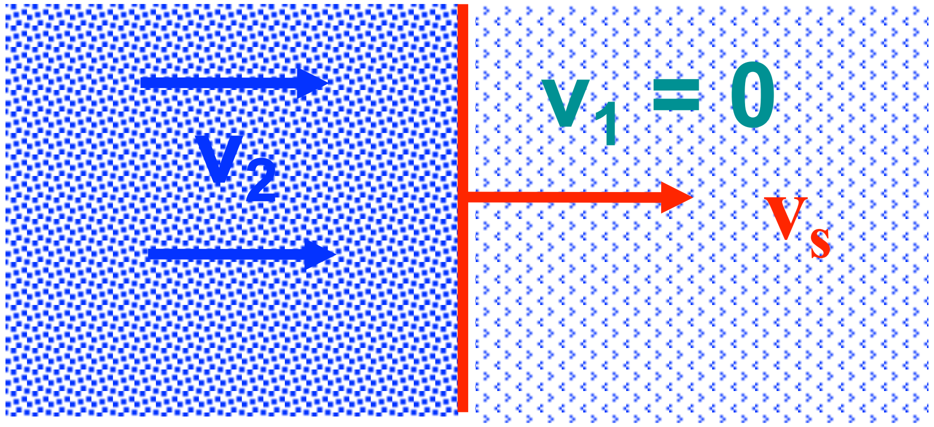
shock discontinuity



compression

heating

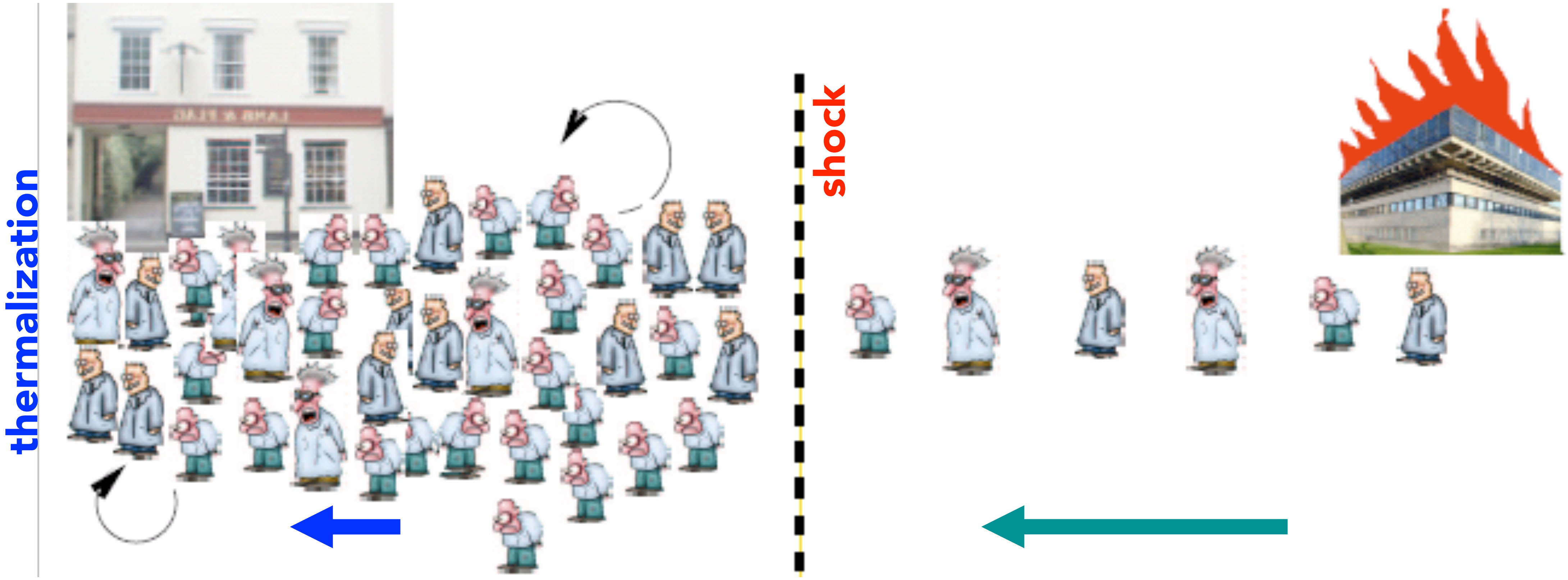
shock frame



upstream frame

$\rho_2\ v_2\ T_2$

$\rho_1\ v_1\ T_1\ p_1$



low bulk speed

compression => high density

much internal motion => high T

high sound speed => subsonic

high bulk speed, supersonic

low density

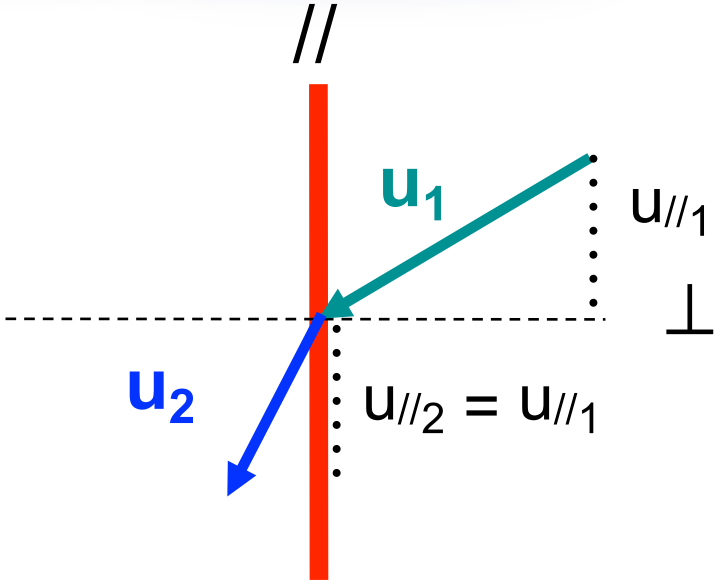
little internal motion => low T

low sound speed

hydro jump conditions

- stationary shock, conservation in the shock frame:
  - ✦ mass $$[\rho u_\perp]_1^2 = 0$$
  - ✦ momentum $$[p + \rho u_\perp^2]_1^2 = 0 \; et \; [\rho u_\perp u_{//}]_1^2 = 0$$
  - ✦ energy $$\left[\left\{\frac{1}{2}\rho(u_\perp^2 + u_{//}^2) + u_{int} + p\right\}.u_\perp\right]_1^2 = 0$$

or $$\left[\left\{\frac{1}{2}\rho(u_\perp^2 + u_{//}^2) + \frac{\gamma p}{\gamma - 1}\right\}.u_\perp\right]_1^2 = 0$$



continuity of u//
⇒ velocity rotates

closer to shock front

- Rankine-Hugoniot (1889): ⊥ and strong shock $p_2 \gg p_1$

Mach number M

compression ratio r

$$r = \frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}$$

$$c_{son}^2 = \gamma p/\rho$$

$$M = v / c_{son}$$

$$p + \rho u^2 = p(1 + \gamma M^2)$$

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{(\gamma + 1)}$$

$$\frac{T_2}{T_1} = \frac{p_2\rho_1}{p_1\rho_2} = \frac{[2\gamma M_1^2 - (\gamma-1)][(\gamma-1)\gamma M_1^2 + 2]}{(\gamma+1)^2 M_1^2}$$

highly supersonic shock: $M_1 \gg 1$

- for a large Mach number: $M_1 \gg 1$

  compression saturates to

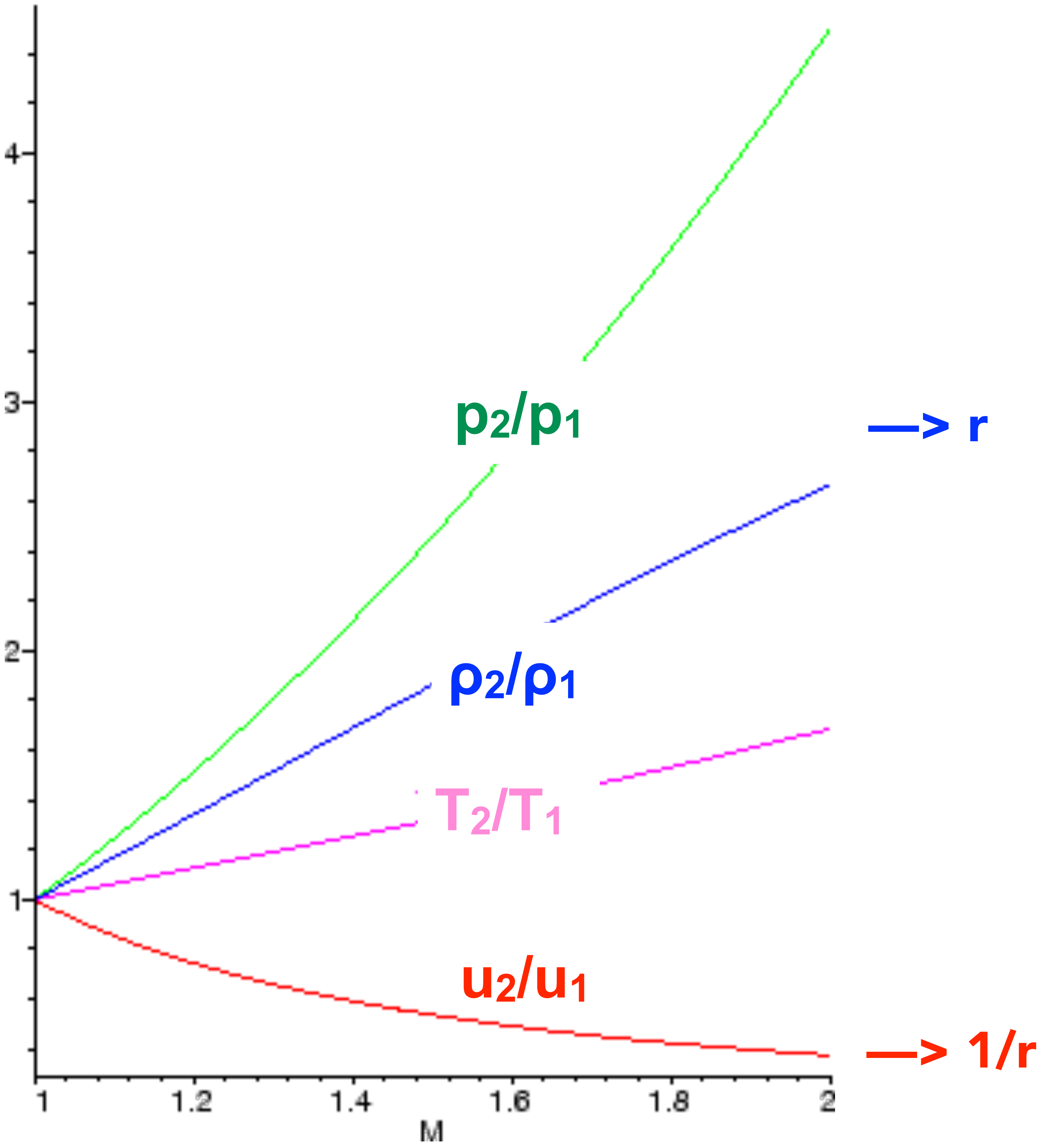
$$\boxed{\mathbf{r} = \frac{\mathbf{u_1}}{\mathbf{u_2}} = \frac{\rho_2}{\mathbf{u_1}} \rightarrow \frac{\gamma+1}{\gamma-1}}$$

✦ ideal gas (γ = 5/3) **r → 4**

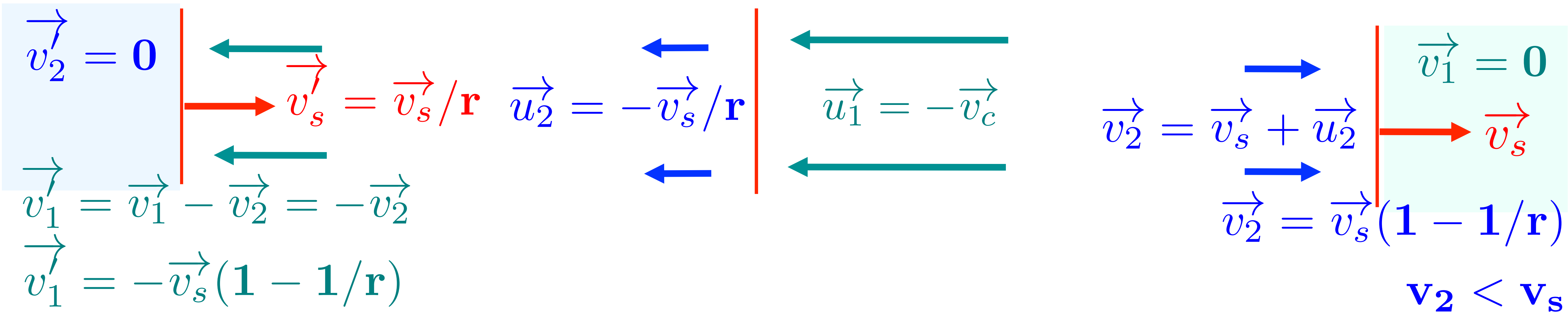
✦ relativistic gas (γ = 4/3) **r → 7**

$$\frac{\mathbf{p_2}}{\mathbf{p_1}} \rightarrow \frac{2\gamma}{\gamma+1}\mathbf{M}_1^2 \uparrow\uparrow\uparrow$$

$$\frac{\mathbf{T_2}}{\mathbf{T_1}} \rightarrow \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\mathbf{M}_1^2 \uparrow\uparrow$$



different points of view



downstream frame

shock frame

upstream frame

$\vec{v_2'} = \mathbf{0}$

$\vec{v_s'} = \vec{v_s}/\mathbf{r}$

$\vec{v_1'} = \vec{v_1} - \vec{v_2} = -\vec{v_2}$

$\vec{v_1'} = -\vec{v_s}(\mathbf{1} - \mathbf{1}/\mathbf{r})$

$\vec{u_2} = -\vec{v_s}/\mathbf{r}$

$\vec{u_1} = -\vec{v_c}$

$\vec{v_1} = \mathbf{0}$

$\vec{v_s}$

$\vec{v_2} = \vec{v_s} + \vec{u_2}$

$\vec{v_2} = \vec{v_s}(\mathbf{1} - \mathbf{1}/\mathbf{r})$

$\mathbf{v_2} < \mathbf{v_s}$

- in the shock frame (or in upstream frame when replacing u_1 with v_s):

  incident ram energy converted to disorder: high pressure and high temperature

$$\mathbf{p_2} \rightarrow \frac{\mathbf{2}\gamma}{\gamma+\mathbf{1}}\mathbf{M_1^2}\,\mathbf{p_1} = \left(\frac{\mathbf{2}}{\gamma+\mathbf{1}}\right)\rho_\mathbf{1}\mathbf{u_1^2} = \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{r}}\right)\rho_\mathbf{1}\mathbf{u_1^2} \quad \text{and} \quad \mathbf{p_{ram2}} = \rho_\mathbf{2}\mathbf{u_2^2} = \frac{\mathbf{1}}{\mathbf{r}}\rho_\mathbf{1}\mathbf{u_1^2}$$

$$\mathbf{T_2} = \frac{\overline{\mathbf{m}}}{\mathbf{k}}\frac{\mathbf{p_2}}{\rho_\mathbf{2}} \rightarrow \frac{\mathbf{2}(\gamma-\mathbf{1})}{(\gamma+\mathbf{1})^\mathbf{2}}\frac{\overline{\mathbf{m}}}{\mathbf{k}}\mathbf{u_1^2} = \frac{(\mathbf{r}-\mathbf{1})}{\mathbf{r^2}}\frac{\overline{\mathbf{m}}}{\mathbf{k}}\mathbf{u_1^2}$$

- subsonic motion downstream => uniform p_2

$$\mathbf{M_2^2} = \frac{(\gamma-\mathbf{1})\mathbf{M_1^2}+\mathbf{2}}{\mathbf{2}\gamma\mathbf{M_1^2}-(\gamma-\mathbf{1})} \rightarrow \frac{(\gamma-\mathbf{1})}{\mathbf{2}\gamma} < \mathbf{1}$$

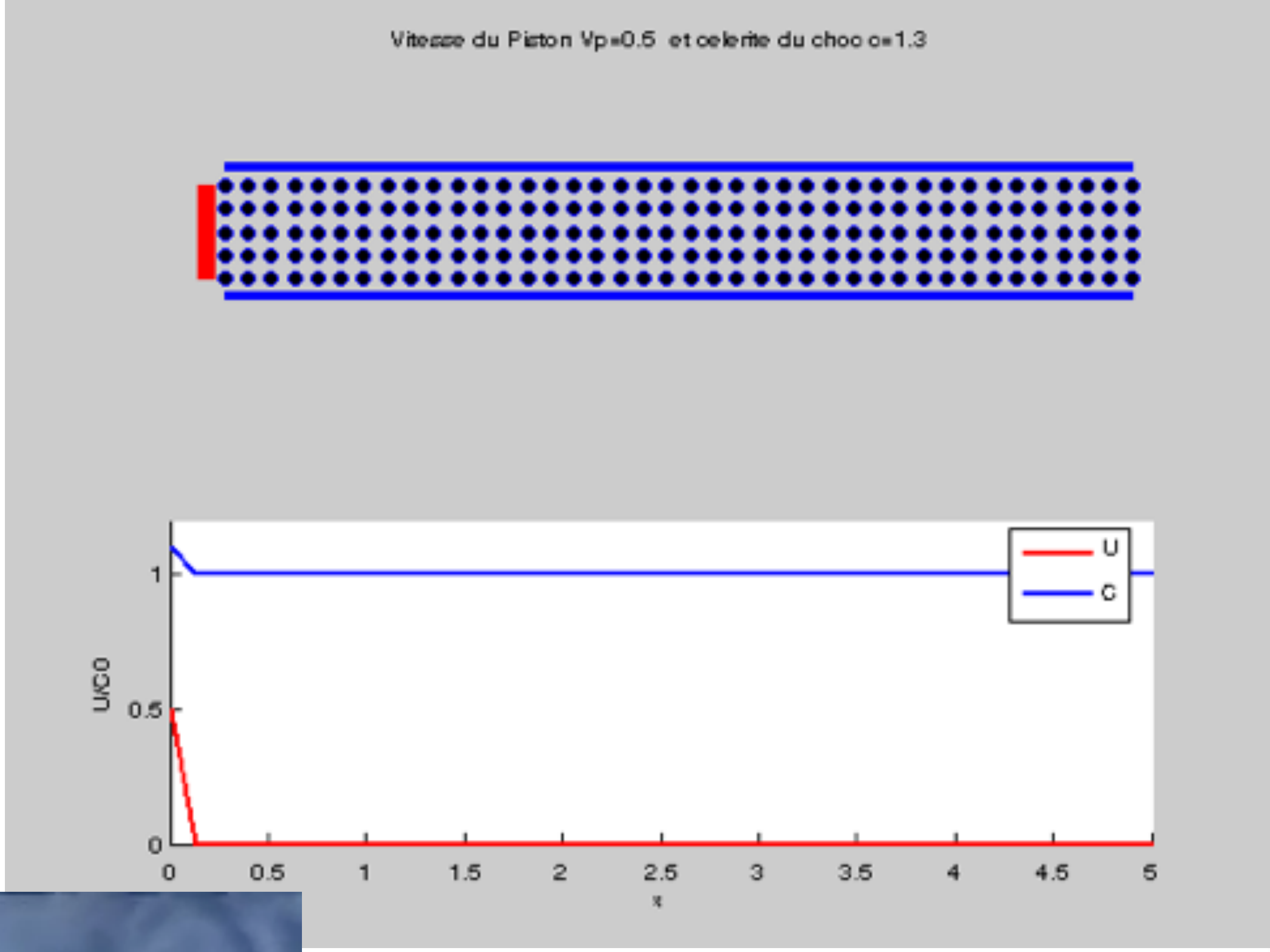
- equipartition in downstream gas ($E_{\text{kin microscopic}} = E_{\text{kin bulk}}$) viewed from the upstream frame

$$\frac{\mathbf{dU_{int2}}}{\mathbf{dm}} = \frac{\mathbf{1}}{\rho_\mathbf{2}}\frac{\mathbf{dU_{int2}}}{\mathbf{dV}} = \frac{\mathbf{3}}{\mathbf{2}}\frac{\mathbf{p_2}}{\rho_\mathbf{2}} = \frac{\mathbf{3}(\gamma-\mathbf{1})}{(\gamma+\mathbf{1})^\mathbf{2}}\mathbf{u_1^2} = \frac{\mathbf{9}}{\mathbf{32}}\mathbf{v_s^2} \quad \text{(frame ind.)}$$

$$\frac{\mathbf{dE_{ram2}}}{\mathbf{dm}} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{v_2^2} = \frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{r}}\right)^\mathbf{2}\mathbf{v_s^2} = \frac{\mathbf{2}}{(\gamma+\mathbf{1})^\mathbf{2}}\mathbf{v_s^2} = \frac{\mathbf{9}}{\mathbf{32}}\mathbf{v_s^2} \quad \left(\neq \frac{\mathbf{1}}{\mathbf{2}}\mathbf{u_2^2}\right)$$

piston shock

- piston with velocity $v_p = 0.5\ c_1$
- pressure shock running ahead of the piston (detached) at velocity $v_s = 1.3\ c_1$

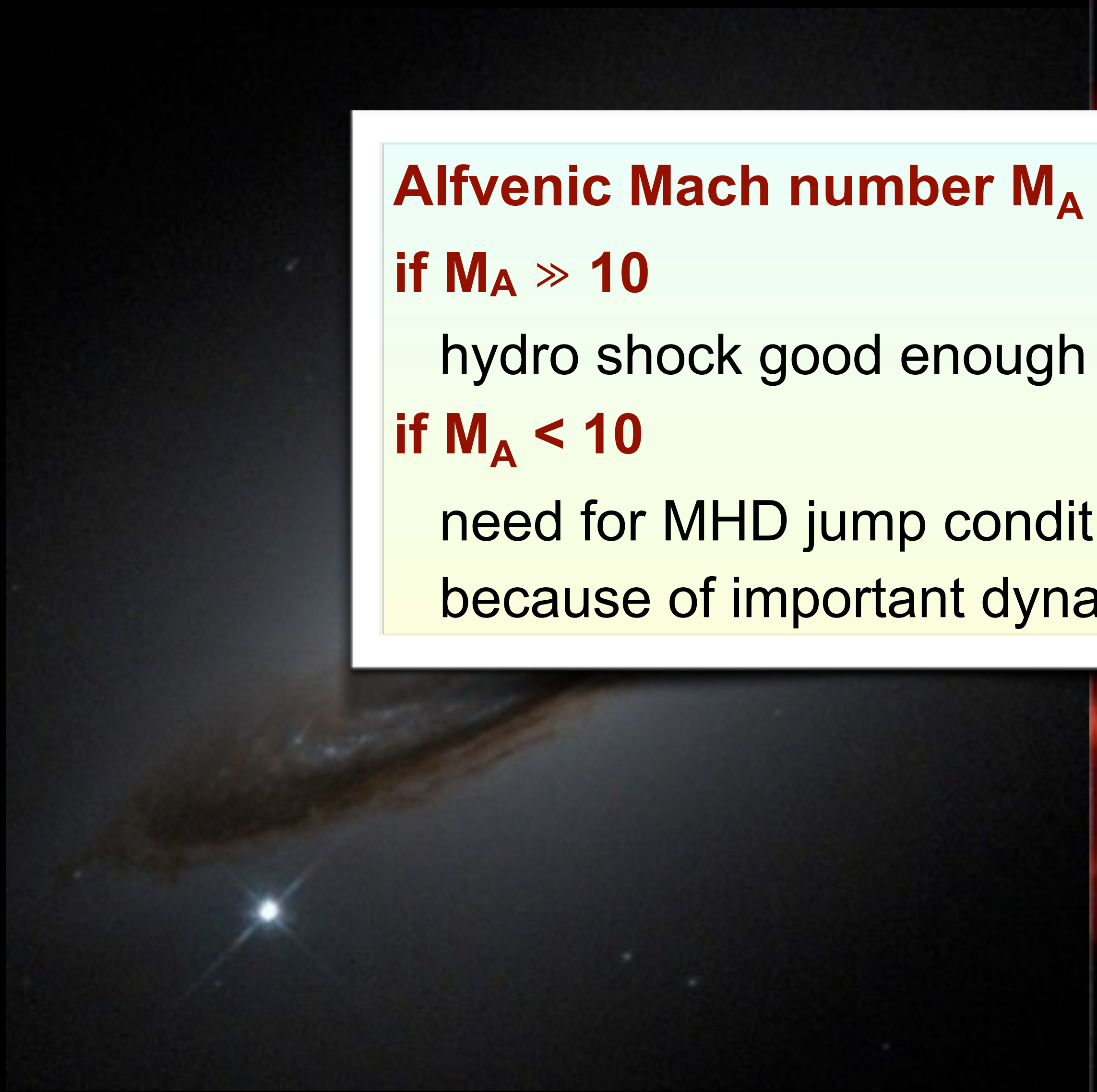


http://ufrmeca.univ-lyon1.fr/~buffat/COURS/AERO_HTML/node49.html

- Eyjafjallaj¨kull (Iceland) 2010

downstream profiles

- ex: upstream gas = HI atomic interstellar cloud at 100 K (c_s = 1 km/s)
  supernova shockwave $v_s = 10^3$ km/s
  $T_2/T_1 \rightarrow 5\ M_1^2/16 \Rightarrow T_2 = 3\ 10^7$ K



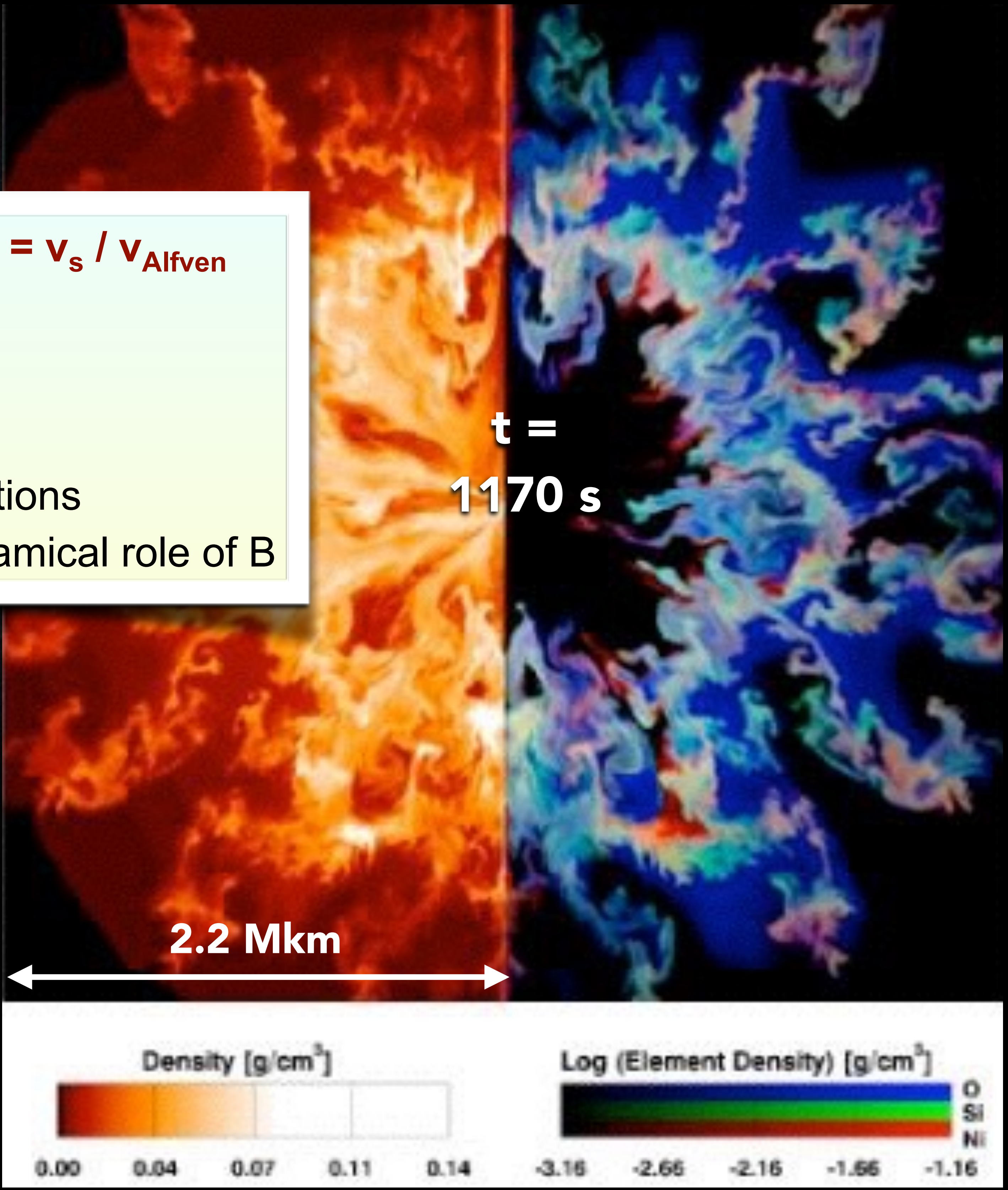
Alfvenic Mach number $M_A = v_s / v_{Alfven}$

if $M_A \gg 10$

hydro shock good enough

if $M_A < 10$

need for MHD jump conditions
because of important dynamical role of B



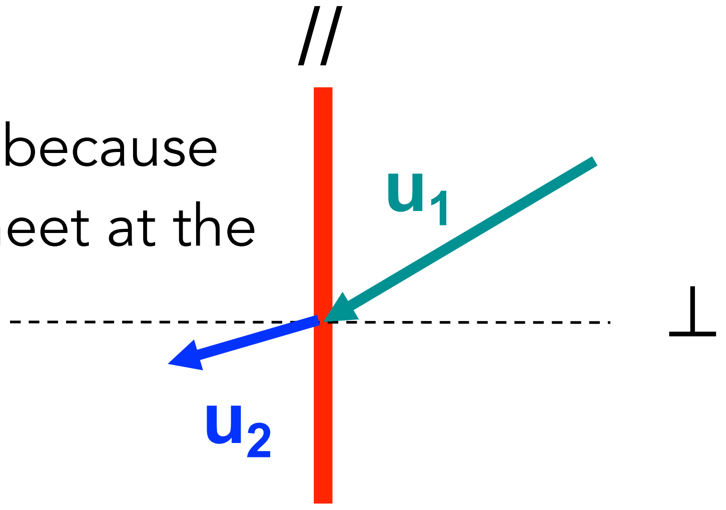
Kifonidis +2000

- momentum imparted to nuclei
  electrons heat up later through
  Coulomb interactions

MHD jump conditions

- conservations (B resists in pressure and in tension)
  - ✦ masse $[\rho u_\perp]_1^2=0$
  - ✦ momentum $\left[p+\rho u_\perp^2+\frac{1}{2\mu_0}(B_{//}^2-B_\perp^2)\right]_1^2=0\,et\,\left[\rho u_\perp u_{//}-\frac{1}{\mu_0}B_{//}B_\perp\right]_1^2=0$
  - ✦ energy $\left[\left\{\frac{1}{2}\rho(u_\perp^2+u_{//}^2)+u_{int}+p\right\}.u_\perp-\frac{1}{\mu_0}\left\{B_\perp u_{//}-B_{//}u_\perp\right\}.B_{//}\right]_1^2=0$
  - ✦ mgn flux $[B_\perp u_{//}-B_{//}u_\perp]_1^2=0\,et\,B_{\perp 1}=B_{\perp 2}$

$u_{//}$ changed because of current sheet at the shock front



- ⊥ shock and strong shock $p_2 \gg p_1$
  - ✦ compression ratio X

$$\boxed{\mathbf{X}=\frac{\mathbf{u_1}}{\mathbf{u_2}}=\frac{\rho_2}{\rho_1}=\frac{\mathbf{B_2}}{\mathbf{B_1}}}$$

  - ✦ solution of

$$\mathbf{2p_{B1}(\gamma-2)X^3-X^2\left[2\gamma p_{th1}+(\gamma-1)(p_{ram1}+4p_{B1})\right]}$$
$$\mathbf{+2\gamma X\left[p_{th1}+p_{ram1}+p_{B1}\right]-(\gamma+1)p_{ram1}=0}$$

  - ✦ if $p_B \rightarrow 0$: X → 4
  - ✦ if high magnetic pressure p_B: **X → 0**